Stochastic differential equations driven by fractional Brownian motions

Fabrice Baudoin[*]

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[*]Department of Mathematics of Purdue University, USA, fbaudoin@math.purdue.edu

1   Fractional Brownian motion

We remind that a stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Gaussian process if for every $t_1, \cdots, t_n \in \mathbb{R}_{\geq 0}$, the random vector $(X_{t_1}, \cdots, X_{t_n})$ is Gaussian. The distribution of a Gaussian process $(X_t)_{t\geq 0}$ is uniquely determined by its mean function

$$m(t) = \mathbb{E}(X_t),$$

and its covariance function

$$R(s, t) = \mathbb{E}((X_t - m(t))(X_s - m(s))).$$

We can observe that the covariance function $R(s, t)$ is symmetric, that is $R(s, t) = R(t, s)$, and positive, that is for $a_1, ..., a_n \in \mathbb{R}$ and $t_1, ..., t_n \in \mathbb{R}_{\geq 0}$,

$$\sum_{1\leq i,j\leq n} a_i a_j R(t_i, t_j) = \sum_{1\leq i,j\leq n} a_i a_j \mathbb{E}\left((X_{t_i} - m(t_i))(X_{t_j} - m(t_j))\right)$$

$$= \mathbb{E}\left(\left(\sum_{i=1}^{n}(X_{t_i} - m(t_i))\right)^2\right) \geq 0.$$

As an application of the Daniell-Kolmogorov theorem, it is possible to prove the following basic existence result for Gaussian processes.

Proposition 1.1 *Let $m : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and let $R : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a symmetric and positive function. There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian process $(X_t)_{t\geq 0}$ defined on it, whose mean function is m and whose covariance function is R.*

Definition 1.2 *Let $H \in (0, 1]$. A continuous Gaussian process $(B_t)_{t\geq 0}$ is called a fractional Brownian motion with Hurst parameter H if its mean function is 0 and its covariance function is*

$$R(s, t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right)$$

We may observe that for $H = \frac{1}{2}$, the covariance function becomes

$$R(s, t) = \min(s, t).$$

As a consequence, a fractional Brownian motion with Hurst parameter $H = \frac{1}{2}$ is a Brownian motion. If $(B_t)_{t\geq 0}$ is a fractional Brownian motion with parameter H, then we have

$$\mathbb{E}(B_t - B_s) = 0,$$

and
$$\mathbb{E}((B_t - B_s)^2) = \mathbb{E}(B_t^2) + \mathbb{E}(B_s^2) - 2\mathbb{E}(B_t B_s) = |t - s|^{2H}.$$

We can deduce from this simple computation that fractional has stationary increments: For every $h > 0$, the two processes $(B_{t+h} - B_h)_{t>0}$ and $(B_t)_{t\geq 0}$ have the same distribution. Fractional Brownian motion also enjoys a scaling invariance property. Indeed, for $c > 0$, we have

$$\mathbb{E}(B_{ct} B_{cs}) = c^{2H} \mathbb{E}(B_t B_s).$$

Therefore, for every $c > 0$, the two processes $(B_{ct})_{t\geq 0}$ and $(c^H B_t)_{t\geq 0}$ have the same distribution. Up to a constant, fractional Brownian motion is actually the only continuous Gaussian process that enjoys the two above properties. We let the proof of the following proposition as an exercise.

Proposition 1.3 *Let $(X_t)_{t\geq 0}$ be a continuous Gaussian process with stationary increments such that for every $c > 0$, the two processes $(X_{ct})_{t\geq 0}$ and $(c^H X_t)_{t\geq 0}$ have the same distribution. Then, there is a constant σ, such that $X_t = \sigma B_t$, where $(B_t)_{t\geq 0}$ is a fractional Brownian motion with parameter H.*

We now turn to the regularity of fractional Brownian motion paths. In the theory of stochastic processes, we often use the Hölder scale to quantify the regularity of the paths of a process. Let $\gamma \in [0,1]$. A function $f : [0,T] \to \mathbb{R}$ is said to be γ-Hölder continuous if there exists a constant $c \geq 0$ such that for every $s, t \in [0,t]$,

$$|f(t) - f(s)| \leq c|t - s|^{\gamma}.$$

The most useful tool to study the Hölder regularity of stochastic processes is certainly the celebrated Kolmogorov continuity theorem.

Theorem 1.4 *(Kolmogorov continuity theorem) Let $\alpha, \varepsilon, c > 0$. If a process $(X_t)_{t\in[0,T]}$ satisfies for $s, t \in [0,T]$,*

$$\mathbb{E}\left(|X_t - X_s|^{\alpha}\right) \leq c \mid t - s \mid^{1+\varepsilon},$$

then there exists a modification of the process $(X_t)_{t\in[0,T]}$ that is a continuous process and whose paths are γ-Hölder continuous for every $\gamma \in [0, \frac{\varepsilon}{\alpha})$.

As a consequence of the Kolmogorov continuity theorem we deduce:

Proposition 1.5 *Let $(B_t)_{t\geq 0}$ be a fractional Brownian motion with parameter H. For every $T > 0$ and $0 < \varepsilon < H$, there is a finite random variable $\eta_{\varepsilon,T}$ such that for every $s, t \in [0,T]$,*

$$|B_t - B_s| \leq \eta_{\varepsilon,T}|t - s|^{H-\varepsilon}, \quad a.s.$$

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Proof. As seen before, we have

$$\mathbb{E}((B_t - B_s)^2) = \mathbb{E}(B_t^2) + \mathbb{E}(B_s^2) - 2\mathbb{E}(B_t B_s) = |t - s|^{2H}.$$

More generally, due to the stationarity of the increments and the scaling property of fractional Brownian motion, we have for every $k \geq 1/H$,

$$\mathbb{E}(|B_t - B_s|^k) = \mathbb{E}(|B_{t-s}|^k) = |t - s|^{Hk}\mathbb{E}(|B_1|^k).$$

As a consequence of the Kolmogorov continuity theorem we deduce that there is a modification $(\tilde{B}_t)_{t \in [0,T]}$ of the process $(B_t)_{t \in [0,T]}$ that is a continuous process and whose paths are γ-Hölder continuous for every $\gamma \in [0, H - \frac{1}{k})$. Since $(\tilde{B}_t)_{t \in [0,T]}$ and $(B_t)_{t \in [0,T]}$ are both continuous, we deduce that these two processes are actually the same. $\qquad\square$

As a consequence of a result in [1], if $(B_t)_{t \geq 0}$ is a fractional Brownian motion with parameter H, then

$$\mathbb{P}\left(\limsup_{u \to 0} \frac{B_u}{u^H\sqrt{\ln\ln u^{-1}}} = 1\right) = 1.$$

Hence, the paths of a fractional Brownian motion are not γ-Hölder for any $\gamma \geq H$. It is sometime useful to notice that fractional Brownian motion can actually be constructed from the Brownian motion by using Wiener integrals.

Namely, if $(\beta_t)_{t \geq 0}$ is a Brownian motion, then the process

$$B_t = \int_0^t K_H(t, s)d\beta_s, t \geq 0 \tag{1.1}$$

is a fractional Brownian motion with Hurst parameter H, where, if $H > \frac{1}{2}$,

$$K_H(t, s) = c_H s^{\frac{1}{2} - H} \int_s^t (u - s)^{H - \frac{3}{2}} u^{H - \frac{1}{2}} du, \qquad t > s.$$

and c_H is a suitable normalization constant. When $H < \frac{1}{2}$, the expression for $K_H(t, s)$ is more difficult and we refer the interested reader to [3].

2 Young's integrals and stochastic differential equations driven by fractional Brownian motions

2.1 Young's integral and basic estimates

One of the main objectives in this course is to study solutions of stochastic differential equations that can be writen as

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

where $(B_t)_{t \geq 0}$ is a fractional Brownian motion with parameter H. A first step is to understand what should be the correct meaning of the integral $\int \sigma(X_t) dB_t$ that indirectly appears in this equation. Since, for $H < 1$, $(B_t)_{t \geq 0}$ does not have absolutely continuous paths, we can not directly use the theory of Riemann-Stieltjes integrals to give a sense to integrals like $\int_0^t f(s) dB_s$ for every continuous functions f. However, as it has understood by L.C Young [8], if f is regular enough in the Hölder sense, then $\int_0^t f(s) dB_s$ can still be constructed as a limit of Riemann sums. In the sequel, we shall denote by $\mathcal{C}^\alpha(I)$ the space of α- Hölder continuous functions that are defined on an interval I.

The basic result of L.C. Young is the following:

Theorem 2.1 *Let $f \in \mathcal{C}^\beta([0, T])$ and $g \in \mathcal{C}^\gamma([0, T])$. If $\beta + \gamma > 1$, then for every subdivision t_i^n of $[0, T]$, whose mesh tends to 0, the Riemann sums*

$$\sum_{i=0}^{n-1} f(t_i^n)(g(t_{i+1}^n) - g(t_i^n))$$

converge, when $n \to \infty$ to a limit which is independent of the subdivision t_i^n. This limit is denoted $\int_0^T f dg$ and called the Young's integral of f with respect to g.

This theorem is the cornerstone of the theory of stochastic differential equations driven by a fractional Brownian motion when $H > 1/2$. Indeed, intuitively, the solution of

$$dX_t = b(X_t) dt + \sigma(X_t) dB_t,$$

should have the same Hölder regularity as $(B_t)_{t \geq 0}$ that is $H - \varepsilon$ Hölder. Thus, provided for instance that σ is Lipschitz continuous, the integral $\int \sigma(X_t) dB_t$ is well defined as a Young integral when $H > 1/2$.

The case $H = 1/2$, corresponds to the Brownian motion case. In that case the integrals are understood as Itô's. By using rough paths theory, it is actually possible to define and study stochastic differential equations driven by a fractional Brownian motion when $H > 1/4$. But rough paths theory is beyond the scope of this graduate level course and, in the sequel, we will always restrict ourselves to the case $H > 1/2$.

For later use we need to study how to bound Young's integrals. In the sequel, we fix $T > 0$ and endow the space $\mathcal{C}^\alpha([0, T])$ with the norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\alpha},$$

where, as usual, $\|f\|_\infty = \sup_{t \in [0,T]} |f(t)|$.

For Young's integrals, we have the following basic estimates:

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Proposition 2.2 *Let $f \in \mathcal{C}^\beta([0,T])$ and $g \in \mathcal{C}^\gamma([0,T])$ with $\beta + \gamma > 1$. For every $0 \le s \le t \le T$, we have:*

$$\left| \int_s^t f dg \right| \le K \|f\|_\beta (t-s)^\gamma \left(1 + \frac{(t-s)^\beta}{2^{\beta+\gamma} - 2} \right),$$

where $K = \sup_{0 \le s < t \le T} \frac{|g(t)-g(s)|}{|t-s|^\gamma}$.

The basic tool to prove this estimate is the following representation of the Young's integral which is due to A.A. Ruzmaikina [7].

Lemma 2.3 *$f \in \mathcal{C}^\beta([s,t])$ and $g \in \mathcal{C}^\gamma([s,t])$ with $\beta + \gamma > 1$. Let $s_i^n = s + (t-s)\frac{i}{2^n}$, $0 \le i \le 2^n$ be the dyadic subdivision of $[s,t]$. We have*

$$\int_s^t f dg = f(s)(g(t) - g(s)) + \sum_{k=1}^{+\infty} \sum_{i=0}^{2^{k-1}-1} \left(f(s_{2i+1}^k) - f(s_{2i}^k) \right) \left(g(s_{2i+2}^k) - g(s_{2i+1}^k) \right).$$

Proof. Let us consider the Riemann sum

$$S_n = \sum_{i=0}^{2^n-1} f(s_i^n)(g(s_{i+1}^n) - g(s_i^n)).$$

It is easily seen that

$$S_n = S_{n-1} + \sum_{i=0}^{2^{n-1}-1} \left(f(s_{2i+1}^n) - f(s_{2i}^n) \right) \left(g(s_{2i+2}^n) - g(s_{2i+1}^n) \right).$$

We obtain therefore by induction

$$S_n = f(s)(g(t) - g(s)) + \sum_{k=1}^{n} \sum_{i=0}^{2^{k-1}-1} \left(f(s_{2i+1}^k) - f(s_{2i}^k) \right) \left(g(s_{2i+2}^k) - g(s_{2i+1}^k) \right)$$

and the result follows by letting $n \to \infty$. $\qquad \square$

We can now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2.
We have

$$\left| \int_s^t f dg \right| \le |f(s)(g(t) - g(s))| + \sum_{k=1}^{+\infty} \sum_{i=0}^{2^{k-1}-1} \left| f(s_{2i+1}^k) - f(s_{2i}^k) \right| \left| g(s_{2i+2}^k) - g(s_{2i+1}^k) \right|$$

$$\le K \|f\|_\infty (t-s)^\gamma + K \|f\|_\beta \sum_{k=1}^{+\infty} \sum_{i=0}^{2^{k-1}-1} \frac{(t-s)^\beta}{2^{k\beta}} \frac{(t-s)^\gamma}{2^{k\gamma}}$$

$$\le K \|f\|_\beta (t-s)^\gamma \left(1 + \frac{(t-s)^\beta}{2^{\beta+\gamma} - 2} \right).$$

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The next proposition is proved in the very same way.

Proposition 2.4 *Let $f \in \mathcal{C}^\beta([0,T])$ and $g \in \mathcal{C}^\gamma([0,T])$ with $\beta + \gamma > 1$. For every $0 \le s \le t \le T$, we have:*

- $\sup_{u \in [s,t]} \left| \int_s^u f(v)dv \right| + \sup_{s \le u_1 < u_2 \le t} \frac{\left| \int_{u_1}^{u_2} f(v)dv \right|}{|u_1 - u_2|^\beta} \le \|f\|_\infty (t-s)^{1-\beta}(1 + (t-s)^\beta),$

- *If $\gamma \ge \beta$,*

$$\sup_{u \in [s,t]} \left| \int_s^u f dg \right| + \sup_{s \le u_1 < u_2 \le t} \frac{\left| \int_{u_1}^{u_2} f dg \right|}{|u_1 - u_2|^\beta} \le K \|f\|_\beta (t-s)^{\gamma - \beta}(1 + (t-s)^\beta)\left(1 + \frac{(t-s)^\beta}{2^{\beta+\gamma} - 2}\right)$$

,

where $K = \sup_{0 \le s < t \le T} \frac{|g(t) - g(s)|}{|t-s|^\gamma}$.

2.2 Stochastic differential equations driven by a Hölder path

Our goal in this section is to prove the following theorem.

Theorem 2.5 *Let $g \in \mathcal{C}^\gamma([0,T])$ where $\frac{1}{2} < \gamma \le 1$. Let $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ be two functions such that:*

- *b and σ are globally Lipschitz continuous;*

- *σ is continuously differentiable with a globally Lipschitz derivative.*

For every $x_0 \in \mathbb{R}$, the ordinary differential equation

$$x(t) = x_0 + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dg(s),$$

has a unique solution in $\mathcal{C}^\gamma([0,T])$.

It suffices to prove that for every $x_0 \in \mathbb{R}$ and $\gamma > \beta > 1 - \gamma$, the ordinary differential equation

$$x(t) = x_0 + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dg(s),$$

has a unique solution in $\mathcal{C}^\beta([0,T])$ and, then, that, actually, $x \in \mathcal{C}^\gamma([0,T])$ which is clear from Proposition 2.2. As it is usual in this type of problems , the idea is to use a fixed point argument. In the sequel, we work under the assumptions of

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Theorem 2.5 and chose $\gamma > \beta > 1 - \gamma$. We denote $L = \sup_{0 \le s < t \le T} \frac{|g(t) - g(s)|}{|t-s|^\gamma}$. For $x \in \mathcal{C}^\beta([0, T])$, we denote

$$\Psi(x)(t) = x_0 + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dg(s).$$

For $K > 0$, we define $\mathcal{C}_K^\beta([0, T])$ to be the closed subspace of $\mathcal{C}^\beta([0, T])$ defined by

$$\mathcal{C}_K^\beta([0, T], x_0) = \{f \in \mathcal{C}^\beta([0, T]), f(0) = x_0, |f(t) - f(s)| \le K|t - s|^\beta\}.$$

A first result is the following.

Lemma 2.6 *If $\varepsilon > 0$ is small enough, then Ψ sends $\mathcal{C}_K^\beta([0, \varepsilon], x_0)$ into itself.*

Proof. Let $\varepsilon >, 0 \le s \le t \le \varepsilon$ and $x \in \mathcal{C}_K^\beta([0, \varepsilon], x_0)$. We have

$$|\Psi(x)(t) - \Psi(x)(s)| = \left| \int_s^t b(x(u))du + \int_s^t \sigma(x(u))dg(u) \right|$$

$$\le \left| \int_s^t b(x(u))du \right| + \left| \int_s^t \sigma(x(u))dg(u) \right|$$

We have

$$\left| \int_s^t b(x(u))du \right| \le (t - s) \sup_{t \in [0,\varepsilon]} |b(x(t))|$$

Since b is Lipschitz, we have for some $C \ge 0$,

$$\sup_{t \in [0,\varepsilon]} |b(x(t))| \le |b(x_0)| + CK\varepsilon^\beta.$$

This yields

$$\left| \int_s^t b(x(u))du \right| \le (t - s) \left(|b(x_0)| + CK\varepsilon^\beta \right).$$

On the other hand, from Proposition 2.2, we have

$$\left| \int_s^t \sigma(x(u))dg(u) \right| \le L \left(\sup_{u \in [0,\varepsilon]} |\sigma(x(u))| + \sup_{0 \le u_1 < u_2 \le \varepsilon} \frac{|\sigma(x(u_2)) - \sigma(x(u_1))|}{|u_2 - u_1|^\beta} \right) (t-s)^\gamma \left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2} \right).$$

Using now the fact that σ is Lipschitz, we obtain that for some $C \ge 0$,

$$\sup_{u \in [0,\varepsilon]} |\sigma(x(u))| + \sup_{0 \le u_1 < u_2 \le \varepsilon} \frac{|\sigma(u_2) - \sigma(u_1)|}{|u_2 - u_1|^\beta} \le |\sigma(x_0)| + CK\varepsilon^\beta + CK$$

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As a conclusion we obtain

$$|\Psi(x)(t) - \Psi(x)(s)|$$
$$\leq (t-s)\left(|b(x_0)| + CK\varepsilon^\beta\right) + (t-s)^\gamma \left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\left(|\sigma(x_0)| + CK\varepsilon^\beta + CK\right).$$

This implies,

$$\frac{|\Psi(x)(t) - \Psi(x)(s)|}{(t-s)^\beta}$$
$$\leq (t-s)^{1-\beta}\left(|b(x_0)| + CK\varepsilon^\beta\right) + (t-s)^{\gamma-\beta}\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\left(|\sigma(x_0)| + CK\varepsilon^\beta + CK\right)$$
$$\leq \varepsilon^{1-\beta}\left(|b(x_0)| + CK\varepsilon^\beta\right) + \varepsilon^{\gamma-\beta}\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\left(|\sigma(x_0)| + CK\varepsilon^\beta + CK\right).$$

It is then clear that for ε small enough, we have

$$\varepsilon^{1-\beta}\left(|b(x_0)| + CK\varepsilon^\beta\right) + \varepsilon^{\gamma-\beta}\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\left(|\sigma(x_0)| + CK\varepsilon^\beta + CK\right) \leq K.$$

$$\square$$

The second step is to prove the contraction property

Lemma 2.7 *If $\varepsilon > 0$ is small enough, then Ψ is a contraction on $\mathcal{C}_K^\beta([0,\varepsilon], x_0)$, for the norm*
$$\|f\|_{\beta,\varepsilon} = \sup_{0 \leq s \leq \varepsilon} |f(s)| + \sup_{0 \leq s < t \leq \varepsilon} \frac{|f(t) - f(s)|}{|t-s|^\beta},$$

Proof. In the sequel, we denote by C a common upper bound on the Lipschitz constants of b, σ and σ'. Let $x, y \in \mathcal{C}_K^\beta([0,\varepsilon], x_0)$. From the triangle inequality, we have

$$\|\Psi(x) - \Psi(y)\|_{\beta,\varepsilon} \leq \left\|\int_0^\cdot (b(x(s)) - b(y(s)))ds\right\|_{\beta,\varepsilon} + \left\|\int_0^\cdot (\sigma(x(s)) - \sigma(y(s)))dg(s)\right\|_{\beta,\varepsilon}.$$

From Proposition 2.4, we find

$$\left\|\int_0^\cdot (b(x(s)) - b(y(s)))ds\right\|_{\beta,\varepsilon} \leq C\varepsilon^{1-\beta}(1 + \varepsilon^\beta)\|x - y\|_{\beta,\varepsilon}.$$

Again, from Proposition 2.4,

$$\left\|\int_0^\cdot (\sigma(x(s)) - \sigma(y(s)))dg(s)\right\|_{\beta,\varepsilon} \leq L\varepsilon^{\gamma-\beta}(1 + \varepsilon^\beta)\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\|\sigma(x) - \sigma(y)\|_{\beta,\varepsilon}.$$

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So, we are let to find an upper bound for $\|\sigma(x) - \sigma(y)\|_{\beta,\varepsilon}$. First, it is obvious that

$$\sup_{0 \leq s \leq \varepsilon} |\sigma(x(s)) - \sigma(y(s))| \leq C \sup_{0 \leq s \leq \varepsilon} |x(s) - y(s)|. \leq C\|x - y\|_{\beta,\varepsilon}.$$

Then, we need to control the Hölder regularity of $\sigma(x) - \sigma(y)$. The idea is to write

$$\sigma(x(t)) - \sigma(y(t)) = (x(t) - y(t)) \int_0^1 \sigma'(\alpha x(t) + (1 - \alpha)y(t))d\alpha$$

and

$$\sigma(x(s)) - \sigma(y(s)) = (x(s) - y(s)) \int_0^1 \sigma'(\alpha x(s) + (1 - \alpha)y(s))d\alpha.$$

We have therefore

$$|\sigma(x(t)) - \sigma(y(t)) - (\sigma(x(s)) - \sigma(y(s)))|$$
$$\leq \left| (x(t) - y(t)) \int_0^1 \sigma'(\alpha x(t) + (1 - \alpha)y(t))d\alpha - (x(s) - y(s)) \int_0^1 \sigma'(\alpha x(s) + (1 - \alpha)y(s))d\alpha \right|$$
$$\leq C(t - s)^\beta \|x - y\|_{\beta,\varepsilon} + C|x(s) - y(s)| \int_0^1 \alpha|x(t) - x(s)| + (1 - \alpha)|y(t) - y(s)|)d\alpha$$
$$\leq C(t - s)^\beta \|x - y\|_{\beta,\varepsilon} + KC(t - s)^\beta \|x - y\|_{\beta,\varepsilon}$$
$$\leq (C + KC)(t - s)^\beta \|x - y\|_{\beta,\varepsilon}$$

As a conclusion, we have

$$\|\Psi(x) - \Psi(y)\|_{\beta,\varepsilon} \leq C\varepsilon^{1-\beta}(1+\varepsilon^\beta)\|x-y\|_{\beta,\varepsilon} + L\varepsilon^{\gamma-\beta}(1+\varepsilon^\beta)\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)(C+KC)\|x-y\|_{\beta,\varepsilon},$$

and it is clear that for ε small enough we get a contraction. $\qquad\square$

We are now ready to finish the proof of Theorem 2.5. Indeed, the previous Lemma proves that there is a unique solution to the equation on a small interval $[0, \varepsilon]$, where ε is small enough. More precisely, the estimates have shown that ε should satisfy

$$C\varepsilon^{1-\beta}(1 + \varepsilon^\beta) + L\varepsilon^{\gamma-\beta}(1 + \varepsilon^\beta)\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)(C + KC) < 1,$$

$$\varepsilon^{1-\beta}\left(|b(x_0)| + CK\varepsilon^\beta\right) + \varepsilon^{\gamma-\beta}\left(1 + \frac{\varepsilon^\beta}{2^{\beta+\gamma} - 2}\right)\left(|\sigma(x_0)| + CK\varepsilon^\beta + CK\right) \leq K.$$

Up to the transformation $x(t) = \tan y(t)$ we can assume that b and σ are bounded. In that case, we can then repeat the argument to construct a unique solution on the interval $[\varepsilon, 2\varepsilon]$ and continue like this to finally obtain a globally defined solution.

2.3 Multidimensional extension

The results of the previous sections may be extended to a higher dimensional setting. The following theorem is shown exactly as Theorem 2.5.

Theorem 2.8 *Let* $g \in \mathcal{C}^\gamma([0,T], \mathbb{R}^d)$ *where* $\frac{1}{2} < \gamma \leq 1$. *Let* $b : \mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times d}$ *be two functions such that:*

- *b and σ are globally Lipschitz continuous;*

- *σ is continuously differentiable with a globally Lipschitz derivative.*

For every $x_0 \in \mathbb{R}^n$, the ordinary differential equation

$$x(t) = x_0 + \int_0^t b(x(s))ds + \int_0^t \sigma(x(s))dg(s),$$

has a unique solution in $\mathcal{C}^\gamma([0,T])$ which is valued in $\mathbb{R}^n$.

2.4 Fractional calculus

Another way to handle Young's integrals is to use the so-called fractional calculus. For further details, we refer the reader to ([9]) or ([5]). Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional integrals of f of order α are defined by:

$$I_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^\alpha f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ is the Gamma function. Let us denote by $I_{a+}^\alpha(L^p)$ (respectively $I_{b-}^\alpha(L^p)$) the image of $L^p(a,b)$ by the operator I_{a+}^α (respectively I_{a+}^α). If $f \in I_{a+}^\alpha(L^p)$ (respectively $f \in I_{b-}^\alpha(L^p)$) and $0 < \alpha < 1$, we define for $x \in (a,b)$ the left and right Weyl derivatives by:

$$D_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

and respectively,

$$D_{b-}^\alpha f(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right) \mathbf{1}_{(a,b)}(x)$$

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We have the following property:

$$D_{a+}^{\alpha} D_{a+}^{\beta} = D_{a+}^{\alpha+\beta}, \quad D_{b-}^{\alpha} D_{b-}^{\beta} = D_{b-}^{\alpha+\beta},$$

and for $f \in I_{a+}^{\alpha}(L^p)$, $g \in I_{b-}^{\alpha}(L^p)$

$$\int_a^b D_{a+}^{\alpha} f(t) g(t) dt = (-1)^{-\alpha} \int_a^b f(t) D_{b-}^{\alpha} g(t) dt$$

The key point that allows to use fractional calculus to study Young's integrals is the following Proposition that is due to M. Zähle.

Proposition 2.9 *Let $f \in C^{\lambda}([a,b])$ and $g \in C^{\beta}([a,b])$ with $\lambda + \beta > 1$. Let $1 - \beta < \alpha < \lambda$. Then the Young's integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt,$$

where $g_{b-}(t) = g(t) - g(b)$.

Now recall that from ([5]), for a parameter $0 < \alpha < 1/2$, $W_T^{1-\alpha,\infty}(0,T)$ is defined as the space of measurable function $g : [0,T] \to \mathbb{R}$ such that:

$$\|g\|_{1-\alpha,\infty,T} = \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty$$

Clearly, for every $\epsilon > 0$,

$$\mathcal{C}^{1-\alpha+\epsilon}([0,T]) \subset W_T^{1-\alpha,\infty}(0,T) \subset \mathcal{C}^{1-\alpha}([0,T])$$

Moreover, if $g \in W_T^{1-\alpha,\infty}(0,T)$, its restriction to $(0,t)$ belongs to $I_{t-}^{1-\alpha}(L^{\infty}(0,t))$ for every t and

$$\Lambda_{\alpha}(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |(D_{t-}^{1-\alpha} g_{t-})(s)|.$$

$$\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty$$

where $g_{t-}(s) = g(s) - g(t)$. We also denote by $W_0^{\alpha,1}(0,T)$ the space of measurable functions f on $[0,T]$ such that:

$$\|f\|_{\alpha,1} = \int_0^T \frac{f(s)}{s^{\alpha}} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy ds < \infty$$

The restriction of $f \in W_0^{\alpha,1}(0,T)$ to $(0,t)$ belongs to $I_{0+}^{\alpha}(L^1(0,t))$ for all t. Now putting things together, we have:

$$\int_0^t f dg = (-1)^{\alpha} \int_0^t D_{0+}^{\alpha} f(s) D_{t-}^{1-\alpha} g_{b-}(s) ds$$

and

$$\left| \int_0^t f dg \right| \leq \sup_{0<s<t} \left| (D_{t-}^{1-\alpha} g_{t-})(s) \right| \int_0^t \left| (D_{0+}^{\alpha} f)(s) \right| ds$$
$$\leq \Lambda_{\alpha}(g) \|f\|_{\alpha,1}$$

By using the fractional calculus techniques, we obtain the following sharp theorem which is due to Nualart and Răşcanu (see [5]).

Theorem 2.10 *Let $0 < \alpha < 1/2$ be fixed. Let $g \in W^{1-\alpha,\infty}(0,T;\mathbb{R}^d)$. Consider the deterministic differential equation on $\mathbb{R}^n$:*

$$x_t^i = x_0^i + \int_0^t b^i(s,x_s)ds + \sum_{j=1}^d \int_0^t \sigma^{i,j}(s,x_s)dg_s^j, \quad t \in [0,T] \tag{2.2}$$

$i = 1,\cdots,n$ where $x_0 \in \mathbb{R}^n$, and the coefficients $\sigma^{i,j}, b^i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions satisfying the following assumptions with $\rho = 1/\alpha$, $0 < \eta, \delta \leq 1$ and

$$0 < \alpha < \alpha_0 = \min\left\{\frac{1}{2}, \eta, \frac{\delta}{1+\delta}\right\}$$

1. *$\sigma(t,x) = (\sigma^{i,j}(t,x))_{n \times d}$ is differentiable in x, and there exist some constants $0 < \eta, \delta \leq 1$ and for every $N \geq 0$ there exists $M_N > 0$ such that the following properties hold:*

$$\begin{cases} \|\sigma(t,x) - \sigma(t,y)\| \leq M_0 \|x-y\|, \quad x \in \mathbb{R}^n, \forall t \in [0,T] \\ \\ \|\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)\| \leq M_N \|x-y\|^{\delta} \quad \|x\|, \|y\| \leq N, \forall t \in [0,T] \\ \\ \|\sigma(t,x) - \sigma(t,y)\| + \|\partial_{x_i}\sigma(t,x) - \partial_{y_i}\sigma(t,y)\| \leq M_0 \|t-s\|^{\eta}, \forall t, s \in [0,T] \end{cases}$$
$$\tag{2.3}$$

2. *There exists $b_0 \in L^\rho(0, T; \mathbb{R}^n)$, where $\rho \geq 2$, and for every $N \geq 0$ there exists $L_N > 0$ such that the following properties hold:*

$$\begin{cases} \|b(t, x) - b(t, y)\| \leq L_N \|x - y\|, \quad \forall \|x\|, \|y\| \leq N, \forall t \in [0, T] \\ \\ \|b(t, x)\| \leq L_0 \|x\| + b_0(t), \quad \forall x \in \mathbb{R}^d, \forall t \in [0, T] \end{cases} \tag{2.4}$$

Then the differential equation (2.2) has a unique solution $x \in W_0^{\alpha, \infty}(0, T; \mathbb{R}^n)$. Moreover, the solution x is $(1 - \alpha)$-Hölder continuous.

3 Stochastic differential equations driven by fractional Brownian motions

As seen before d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a Gaussian process

$$B_t = (B_t^1, \ldots, B_t^d), \ t \geq 0,$$

where $B^1, \ldots, B^d$ are d independent centered Gaussian processes with covariance function

$$R(t, s) = \frac{1}{2} \left(s^{2H} + t^{2H} - |t - s|^{2H} \right).$$

We have shown that such a process admits a continuous version whose paths are Hölder γ continuous, $\gamma < H$. Therefore if $H > \frac{1}{2}$ and if the V_i's are smooth Lipschitz maps $\mathbb{R}^d \to \mathbb{R}^d$ with Lipschitz derivatives, the stochastic differential equation,

$$\begin{cases} dX_t^x = V_0(X_t^x) dt + \sum_{i=1}^d V_i(X_t^x) \, dB_t^i \\ X_0^x = x \end{cases} \tag{3.5}$$

has a unique solution for every $x \in \mathbb{R}^d$. Our purpose will be to point out some properties of the solution. We shall in particular focus on the problem of the existence of a density with respect to the Lebesgue measure and prove the following theorem.

Theorem 3.1 *Assume that $V_1(x), \cdots, V_d(x)$ form a basis of $\mathbb{R}^d$, then the random variable X_t^x has a density with respect to the Lebesgue measure for every $t > 0$.*

The correct tool to prove this problem is the so-called Malliavin calculus.

3.1 Malliavin calculus with respect to fractional Brownian motion

Let us first state some basic facts about Malliavin calculus with respect to the fractional Brownian motion. We consider the Wiener space of continuous paths:

$$\mathbb{W}^{\otimes d} = \left(\mathcal{C}([0,1], \mathbb{R}^d), (\mathcal{B}_t)_{0 \leq t \leq 1}, \mathbb{P} \right)$$

where:

1. $\mathcal{C}([0,1], \mathbb{R}^d)$ is the space of continuous functions $[0,1] \to \mathbb{R}^d$;

2. $(\beta_t)_{t \geq 0}$ is the coordinate process defined by $\beta_t(f) = f(t)$, $f \in \mathcal{C}([0,1], \mathbb{R}^d)$;

3. $\mathbb{P}$ is the Wiener measure;

4. $(\mathcal{B}_t)_{0 \leq t \leq 1}$ is the ($\mathbb{P}$-completed) natural filtration of $(\beta_t)_{0 \leq t \leq 1}$.

Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0,1]$. We denote by $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product:

$$\langle (\mathbf{1}_{[0,t_1]}, \cdots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \cdots, \mathbf{1}_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^{d} R(t_i, s_i),$$

where R is the covariance function of the fractional Brownian mnotion. For $\phi, \psi \in \mathcal{H}$, we have

$$\langle \phi, \psi \rangle_{\mathcal{H}} = H(2H-1) \int_0^1 \int_0^1 \mid s - t \mid^{2H-2} \langle \phi(s), \psi(s) \rangle_{\mathbb{R}^d} ds dt.$$

It can be shown that $L^{1/H}([0,1], \mathbb{R}^d) \subset \mathcal{H}$ but that $\mathcal{H}$ also contains distributions. A $\mathcal{B}_1$-measurable real valued random variable F is said to be cylindrical if it can be written as

$$F = f \left(\int_0^1 \langle h_s^1, dB_s \rangle, \ldots, \int_0^1 \langle h_s^n, dB_s \rangle \right),$$

where $h^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function. The set of cylindrical random variables is denoted $\mathcal{S}$. The Malliavin derivative of $F \in \mathcal{S}$ is the $\mathbb{R}^d$ valued stochastic process $(\mathbf{D}_t F)_{0 \leq t \leq 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^{n} h^i(t) \frac{\partial f}{\partial x_i} \left(\int_0^1 \langle h_s^1, dB_s \rangle, \ldots, \int_0^1 \langle h_s^n, dB_s \rangle \right).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$\mathbf{D}_{t_1,\ldots,t_k}^k F = \mathbf{D}_{t_1} \ldots \mathbf{D}_{t_k} F.$$

For any $p \geq 1$, the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $L^p\left(\mathcal{C}([0,1], \mathbb{R}^d), \mathcal{H}^{\otimes k}\right)$. We denote by $\mathbb{D}^{k,p}(\mathcal{H})$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left(F^p\right) + \sum_{j=1}^{k} \mathbb{E}\left(\left\|\mathbf{D}^j F\right\|_{\mathcal{H}^{\otimes j}}^p \right) \right)^{\frac{1}{p}},$$

and

$$\mathbb{D}^\infty(\mathcal{H}) = \bigcap_{p \geq 1} \bigcap_{k \geq 1} \mathbb{D}^{k,p}(\mathcal{H}).$$

We then have the following key result:

Theorem 3.2 *Let $F = (F_1, \ldots, F_n)$ be a $\mathcal{B}_1$-measurable random vector such that:*

1. *For every $i = 1, \ldots, n$, $F_i \in \mathbb{D}^{1,2}(\mathcal{H})$;*

2. *The matrix $\Gamma = (\langle \mathbf{D}F^i, \mathbf{D}F^j \rangle_{\mathcal{H}})_{1 \leq i,j \leq n}$ is invertible almost surely.*

Then the law of F has a density with respect to the Lebesgue measure on $\mathbb{R}^n$. If moreover $F \in \mathbb{D}^\infty(\mathcal{H})$ and, for every $p > 1$,

$$\mathbb{E}\left(\frac{1}{|\det \Gamma|^p} \right) < +\infty,$$

then this density is smooth.

Remark 3.3 *The matrix Γ is called the Malliavin matrix of the random vector F.*

3.2 Existence of the density

We now turn to the proof of Theorem 3.1.
Let us denote by Φ the stochastic flow associated with equation (3.5), that is $\Phi_t(x) = X_t^x$. We have:

Lemma 3.4 *The map Φ_t is C^1 and the first variation process defined by*

$$\mathbf{J}_{0 \to t} = \frac{\partial \Phi_t}{\partial x},$$

satisfies the following equation:

$$\mathbf{J}_{0 \to t} = \mathbf{Id}_{\mathbb{R}^d} + \int_0^t DV_0(X_s^x)\mathbf{J}_{0 \to s}ds + \sum_{i=1}^{d} \int_0^t DV_i(X_s^x)\mathbf{J}_{0 \to s}dB_s^i.$$

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Lemma 3.5 *For every* $i = 1, \ldots, d$, $t > 0$, *and* $x \in \mathbb{R}^d$, $X_t^{x,i} \in \mathbb{D}^\infty(\mathcal{H})$, *and*

$$\mathbf{D}_s^j X_t^x = \mathbf{J}_{0 \to t} \mathbf{J}_{0 \to s}^{-1} V_j(X_s), \quad j = 1, \ldots, d, \quad 0 \leq s \leq t,$$

where $\mathbf{D}_s^j X_t^{x,i}$ *is the j-th component of* $\mathbf{D}_s X_t^{x,i}$.

Therefore,

$$\Gamma_1 = \mathbf{J}_{0 \to 1} \int_0^1 \int_0^1 \mathbf{J}_{0 \to u}^{-1} V(X_u^{x_0}) V(X_v^{x_0})^T \left(\mathbf{J}_{0 \to v}^{-1}\right)^T \mid u - v \mid^{2H-2} du dv \mathbf{J}_{0 \to 1}^T,$$

where V denotes the $d \times d$ matrix $(V_1 \ldots V_d)$.
Since $\mathbf{J}_{0 \to 1}$ is almost surely invertible it is enough to check that with probability one, the matrix

$$C_1 = \int_0^1 \int_0^1 \mathbf{J}_{0 \to u}^{-1} V(X_u^{x_0}) V(X_v^{x_0})^T \left(\mathbf{J}_{0 \to v}^{-1}\right)^T \mid u - v \mid^{2H-2} du dv$$

is invertible. Let us now observe that for $x \in \mathbb{R}^d$

$$\langle x, C_1 x \rangle = \sum_{j=1}^d \int_0^1 \int_0^1 \mid u - v \mid^{2H-2} \langle x, (\mathbf{J}_{0 \to u}^{-1} V_j)(x_0) \rangle \langle x, (\mathbf{J}_{0 \to v}^{-1} V_j)(x_0) \rangle du dv.$$

Therefore, if x is in the kernel of C_1, we have

$$\langle x, (\mathbf{J}_{0 \to u}^{-1} V_j)(x_0) \rangle = 0$$

for $u \in [0, 1]$. In particular, for $u = 0$ we obtain

$$\langle x, V_j(x_0) \rangle$$

and thus $x = 0$.

4 Exercises

1. We assume $H > 1/2$.

   (a) For $s \leq t$, compute the double integral $\int_0^t \int_0^s |u - v|^{2H-2} du dv$.

   (b) Deduce that the function

   $$R(s, t) = \frac{1}{2} \left(t^{2H} + s^{2H} - |t - s|^{2H} \right)$$

   is symmetric and positive.

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2. Show that there is no fractional Brownian motion with Hurst parameter $H > 1$.

3. Are the increments of fractional Brownian motion independent if $H \neq \frac{1}{2}$.

4. Let $(X_t)_{t \geq 0}$ be a continuous Gaussian process with stationary increments such that for every $c > 0$, the two processes $(X_{ct})_{t \geq 0}$ and $(c^H X_t)_{t \geq 0}$ have the same distribution. Show that there is a constant σ, such that $X_t = \sigma B_t$, where $(B_t)_{t \geq 0}$ is a fractional Brownian motion with parameter H.

5. Show that if a function $f : [0, T] \to \mathbb{R}$ is γ-Hölder continuous with a parameter $\gamma > 1$, then the function f is a constant function.

6. Let $(B_t)_{t \geq 0}$ be a fractional Brownian motion with parameter H. Show that for every $T > 0$ and $0 < \varepsilon < T$, there is a positive random variable $\eta_{\varepsilon,T}$ such that $\mathbb{E}(\eta_{T,\varepsilon}^p) < \infty$, $p \geq 1$ and such that for every $s, t \in [0, T]$,

$$|B_t - B_s| \leq \eta_{\varepsilon,T}|t - s|^{H-\varepsilon}, \quad a.s.$$

Hint: You may use, without proof, the so-called Garsia-Rademich-Rumsey inequality: Let $p \geq 1$ and $\alpha > p^{-1}$. Then there exists a constant $C_{\alpha,p} > 0$ such that for any continuous function f on $[0, T]$, and for all $t, s \in [0, T]$ one has:

$$|f(t) - f(s)|^p \leq C_{\alpha,p}|t - s|^{\alpha p - 1} \int_0^T \int_0^T \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} dx dy.$$

7. Let $f \in \mathcal{C}^\beta([0, T])$ and $g \in \mathcal{C}^\gamma([0, T])$ with $\beta + \gamma > 1$. Show that for every subdivision t_i^n of $[0, T]$, whose mesh tends to 0, the Riemann sums

$$\sum_{i=0}^{n-1} f(t_{i+1}^n)(g(t_{i+1}^n) - g(t_i^n))$$

converge when $n \to \infty$ to $\int_0^T f dg$.

8. Let $(B_t)_{t \geq 0}$ be a fractional Brownian motion with parameter $H > 1/2$. By using Riemann sums, show that

$$\int_0^t B_s dB_s = \frac{1}{2} B_t^2$$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function and let $x : \mathbb{R} \to \mathbb{R}$ be a γ-Hölder path with $\gamma > 1/2$. Show that

$$f(x(t)) = f(x(0)) + \int_0^t f'(x(s))dx(s).$$

10. Show that the space $\mathcal{C}^\alpha([0,T])$ endowed with the norm

$$\|f\|_\alpha = \|f\|_\infty + \sup_{0 \leq s < t \leq T} \frac{|f(t) - f(s)|}{|t - s|^\alpha}$$

is a Banach space.

11. Give a complete proof of Proposition 2.4.

12. Solve the following stochastic differential equations driven by a fractional Brownian motion $(B_t)_{t \geq 0}$ with parameter $H > 1/2$.

   (a) $dX_t = \alpha X_t dt + dB_t$, $X_0 = x_0$;

   (b) $dX_t = \alpha X_t dt + \mu X_t dB_t$, $X_0 = x_0$;

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