Rough Paths Theory

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CHAPTER 1

An overview of rough paths theory

Let us consider a differential equation that writes

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s),$$

where the V_i's are vector fields on $\mathbb{R}^n$ and where the driving signal $x(t) = (x^1(t), \cdots, x^d(t))$ is a continuous bounded variation path. If the vector fields are Lipschitz continuous then, for any fixed initial condition, there is a unique solution $y(t)$ to the equation. We can see this solution y as a function of the driving signal x. It is an important question to understand for which topology, this function is continuous.

A simple example shows that the topology of uniform convergence is not the correct one here. Indeed, let us consider the differential equation

$$\begin{aligned} y_1(t) &= x_1(t) \\ y_2(t) &= x_2(t) \\ y_3(t) &= -\int_0^t y_2(s)dx_1(s) + \int_0^t y_1(s)dx_2(s) \end{aligned}$$

where

$$x_1(t) = \frac{1}{n}\cos(n^2 t), \quad x_2(t) = \frac{1}{n}\sin(n^2 t).$$

A straightforward computation shows that $y_3(t) = t$. When $n \to \infty$, (x_1, x_2) converges uniformly to 0 whereas, of course, (y_1, y_2, y_3) does not converge to 0. In this framework , a correct topology is given by the topology of convergence in 1-variation on compact sets. To fix the ideas, let us work on the interval $[0, 1]$. The distance in 1-variation between two continuous bounded variation paths $x, \tilde{x} : [0, 1] \to \mathbb{R}^d$ is given by

$$\delta_1(x, \tilde{x}) = \|x(0) - \tilde{x}(0)\| + \sup_{\pi} \sum_{k=0}^{n-1} \|(x(t_{i+1}) - \tilde{x}(t_{i+1})) - (x(t_i) - \tilde{x}(t_i))\|,$$

where the supremum is taken over all the subdivisions

$$\pi = \{0 \le t_1 \le \cdots \le t_n \le 1\}.$$

It is then a fact that is going to be proved in this class that if the V_i's are bounded and if $x^n : [0, 1] \to \mathbb{R}^d$ is a sequence of bounded variation paths that converges in 1-variation to a continuous path x with bounded variation, then the solutions of the differential equations

$$y^n(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y^n(s))dx^{i,n}(s),$$

converge in 1-variation to the solution of

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s).$$

This type of continuity result suggests to use a topology in p-variation, $p \geq 1$, to try to extend the map $x \to y$ to a larger class of driving signals x. More precisely, for $p \geq 1$, let us denote by $\Omega^p(\mathbb{R}^d)$ the closure of the set of continuous with bounded variation paths $x : [0,1] \to \mathbb{R}^d$ with respect to the distance in p-variation which is given by

$$\delta_p(x, \tilde{x}) = \left(\|x(0) - \tilde{x}(0)\|^p + \sup_{\pi} \sum_{k=0}^{n-1} \|(x(t_{i+1}) - \tilde{x}(t_{i+1})) - (x(t_i) - \tilde{x}(t_i))\|^p \right)^{1/p}.$$

We will then prove the following result:

Proposition 0.1. *Let $p > 2$. If $x^n : [0,1] \to \mathbb{R}^d$ is a sequence of bounded variation paths that converges in p-variation to a path $x \in \Omega^p(\mathbb{R}^d)$, then the solutions of the differential equations*

$$y^n(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y^n(s))dx^{i,n}(s),$$

converge in p-variation to some $y \in \Omega^p(\mathbb{R}^d)$. Moreover y is the solution of the differential equation

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s),$$

where the integrals are understood in the sense of Young's integration.

The value $p = 2$ is really a treshold: The result is simply wrong for $p = 2$. The main idea of the rough paths theory is to introduce a much stronger topology than the convergence in p-variation. This topology, that we now explain, is related to the continuity of lifts of paths in free nilpotent Lie groups.

Let $\mathbb{G}_N(\mathbb{R}^d)$ be the free N-step nilpotent Lie group with d generators $X_1, \cdots, X_d$. If $x : [0,1] \to \mathbb{R}^d$ is continuous with bounded variation, the solution x^* of the equation

$$x^*(t) = \sum_{i=1}^{d} \int_0^t X_i(x^*(s))dx^i(s),$$

is called the lift of x in $\mathbb{G}_N(\mathbb{R}^d)$. For $p \geq 1$, let us denote $\Omega^p\mathbb{G}_N(\mathbb{R}^d)$ the closure of the set of lifted paths $x^* : [0,1] \to \mathbb{G}_N(\mathbb{R}^d)$ with respect to the distance in p-variation which is given by

$$\delta_p^N(x^*, y^*) = \sup_{\pi} \left(\sum_{i=1}^{n-1} d_N \left(y^*_{t_i}(x^*_{t_i})^{-1}, y^*_{t_{i+1}}(x^*_{t_{i+1}})^{-1} \right)^p \right)^{\frac{1}{p}},$$

where d_N denotes the Carnot-Carathéodory distance on the group $\mathbb{G}_N(\mathbb{R}^d)$. This is a distance that will be explained in details later. Its main property is that it is homogeneous with respect to the natural dilation of $\mathbb{G}_N(\mathbb{R}^d)$.

Consider now the map $\mathcal{I}$ which associates with a continuous with bounded variation path $x : [0,1] \to \mathbb{R}^d$ the continuous path with bounded variation $y : [0,1] \to \mathbb{R}^d$ that solves the ordinary differential equation

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s).$$

It is clear that there exists a unique map $\mathcal{I}^*$ from the set of continuous with bounded variation lifted paths $[0,1] \to \mathbb{G}_N(\mathbb{R}^d)$ onto the set of continuous with bounded variation lifted paths $[0,1] \to \mathbb{G}_N(\mathbb{R}^n)$ which makes the following diagram commutative

$$\begin{array}{ccc} & \mathcal{I}^* & \\ x^* & \longrightarrow & y^* \\ \uparrow & & \uparrow \\ x & \longrightarrow & y \\ & \mathcal{I} & \end{array}.$$

The fundamental theorem of Lyons is the following:

Theorem 0.2. *If $N \geq [p]$, then in the topology of δ_p^N-variation, there exists a continuous extension of $\mathcal{I}^*$ from $\Omega^p\mathbb{G}_N(\mathbb{R}^d)$ into $\Omega^p\mathbb{G}_N(\mathbb{R}^n)$.*

In particular, we can now give a sense to differential equations driven by some continuous paths with finite p-variation, for any $p \geq 1$. Indeed, let $x : [0,1] \to \mathbb{R}^d$ which is continuous with a fnite p-variation and assume that there exists $x^* \in \Omega^p\mathbb{G}_N(\mathbb{R}^d)$ whose projection onto $\mathbb{R}^d$ is x. The projection onto $\mathbb{R}^d$ of $\mathcal{I}^*(x^*)$ is then understood as being a solution of

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s).$$

An important example of application is given by the case where the driving signal is a Brownian motion $(B(t))_{t\geq 0}$. Brownian motion has a p-finite variation for any $p > 2$ and, as we will see, admits a canonical lift in $\Omega^p\mathbb{G}_2(\mathbb{R}^d)$. As a conclusion, we can consider in the rough paths sense, solutions to the equation

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dB^i(s).$$

It turns out that this notion of solution is exactly equivalent to solutions that are obtained by using the Stratonovitch integration theory. Therefore, the theory of stochastic differential equations appears as a very special case of the rough paths theory.

CHAPTER 2

Ordinary differential equations

1. Continuous paths with bounded variation

The first few lectures are essentially reminders of undegraduate real analysis materials. We will cover some aspects of the theory of differential equations driven by continuous paths with bounded variation. The point is to fix some notations that will be used throughout the course and to stress the importance of the topology of convergence in 1-variation if we are interested in stability results for solutions with respect to the driving signal.

If $s \le t$, we will denote by $\Delta[s,t]$, the set of subdivisions of the interval $[s,t]$, that is $\Pi \in \Delta_[s,t]$ can be written

$$\Pi = \{s = t_0 < t_1 < \cdots < t_n = t\}.$$

Definition 1.1. *A continuous path $x : [s,t] \to \mathbb{R}^d$ is said to have a bounded variation on $[s,t]$, if the 1-variation of x on $[s,t]$, which is defined as*

$$\|x\|_{1-var;[s,t]} := \sup_{\Pi \in \Delta[s,t]} \sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|,$$

is finite. The space of continuous bounded variation paths $x : [s,t] \to \mathbb{R}^d$, will be denoted by $C^{1-var}([s,t],\mathbb{R}^d)$.

$\|\cdot\|_{1-var;[s,t]}$ is not a norm, because constant functions have a zero 1-variation, but it is oviously a semi-norm. If x is continuously differentiable on $[s,t]$, it is easily seen that

$$\|x\|_{1-var,[s,t]} = \int_s^t \|x'(s)\| ds.$$

Proposition 1.2. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. The function $(s,t) \to \|x\|_{1-var,[s,t]}$ is additive, i.e for $0 \le s \le t \le u \le T$,*

$$\|x\|_{1-var,[s,t]} + \|x\|_{1-var,[t,u]} = \|x\|_{1-var,[s,u]},$$

and controls x in the sense that for $0 \le s \le t \le T$,

$$\|x(s) - x(t)\| \le \|x\|_{1-var,[s,t]}.$$

The function $s \to \|x\|_{1-var,[0,s]}$ is moreover continuous and non decreasing.

PROOF. If $\Pi_1 \in \Delta[s,t]$ and $\Pi_2 \in \Delta[t,u]$, then $\Pi_1 \cup \Pi_2 \in \Delta[s,u]$. As a consequence, we obtain

$$\sup_{\Pi_1 \in \Delta[s,t]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\| + \sup_{\Pi_2 \in \Delta[t,u]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\| \le \sup_{\Pi \in \Delta[s,u]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\|,$$

thus

$$\|x\|_{1-var,[s,t]} + \|x\|_{1-var,[t,u]} \le \|x\|_{1-var,[s,u]}.$$

Let now $\Pi \in \Delta[s,u]$:

$$\Pi = \{s = t_0 < t_1 < \cdots < t_n = t\}.$$

Let $k = \max\{j, t_j \le t\}$. By the triangle inequality, we have

$$\begin{aligned}\sum_{j=0}^{n-1} \|x(t_{j+1}) - x(t_j)\| &\le \sum_{j=0}^{k-1} \|x(t_{j+1}) - x(t_j)\| + \sum_{j=k}^{n-1} \|x(t_{j+1}) - x(t_j)\| \\ &\le \|x\|_{1-var,[s,t]} + \|x\|_{1-var,[t,u]}.\end{aligned}$$

Taking the sup of $\Pi \in \Delta[s,u]$ gives

$$\|x\|_{1-var,[s,t]} + \|x\|_{1-var,[t,u]} \ge \|x\|_{1-var,[s,u]},$$

which completes the proof. The proof of the continuity and monoticity of $s \to \|x\|_{1-var,[0,s]}$ is let to the reader. $\square$

This control of the path by the 1-variation norm is an illustration of the notion of controlled path which is very useful in rough paths theory.

Definition 1.3. *A map $\omega : \{0 \le s \le t \le T\} \to [0,\infty)$ is called superadditive if for all $s \le t \le u$,*

$$\omega(s,t) + \omega(t,u) \le \omega(s,u).$$

If, in adition, ω is continuous and $\omega(t,t) = 0$, we call ω a control. We say that a path $x : [0,T] \to \mathbb{R}$ is controlled by a control ω, if there exists a constant $C > 0$, such that for every $0 \le s \le t \le T$,

$$\|x(t) - x(s)\| \le C\omega(s,t).$$

Obviously, Lipschitz functions have a bounded variation. The converse is of course not true: $t \to \sqrt{t}$ has a bounded variation on $[0,1]$ but is not Lipschitz. However, any continuous path with bounded variation is the reparametrization of a Lipschitz path in the following sense.

Proposition 1.4. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. There exist a Lipschitz function $y : [0,1] \to \mathbb{R}^d$, and a continuous and non-decreasing function $\phi : [0,T] \to [0,1]$ such that $x = y \circ \phi$.*

PROOF. We assume $\|x\|_{1-var,[0,T]} \ne 0$ and consider

$$\phi(t) = \frac{\|x\|_{1-var,[0,t]}}{\|x\|_{1-var,[0,T]}}.$$

It is continuous and non decreasing. There exists a function y such that $x = y \circ \phi$ because $\phi(t_1) = \phi(t_2)$ implies $x(t_1) = x(t_2)$. We have then, for $s \le t$,

$$\|y(\phi(t)) - y(\phi(s))\| = \|x(t) - x(s)\| \le \|x\|_{1-var,[s,t]} = \|x\|_{1-var,[0,T]}(\phi(t) - \phi(s)).$$

□

The next result shows that the set of continuous paths with bounded variation is a Banach space.

Theorem 1.5. *The space $C^{1-var}([0,T], \mathbb{R}^d)$ endowed with the norm $\|x(0)\| + \|x\|_{1-var,[0,T]}$ is a Banach space.*

PROOF. Let $x^n \in C^{1-var}([0,T], \mathbb{R}^d)$ be a Cauchy sequence. It is clear that

$$\|x^n - x^m\|_\infty \le \|x^n(0) - x^m(0)\| + \|x^n - x^m\|_{1-var,[0,T]}.$$

Thus, x^n converges uniformly to a continuous path $x : [0,T] \to \mathbb{R}$. We need to prove that x has a bounded variation. Let

$$\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$$

be a a subdivision of $[0,T]$. There is $m \ge 0$, such that $\|x - x^m\|_\infty \le \frac{1}{2n}$, thus

$$\begin{aligned}\sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\| &\le \sum_{k=0}^{n-1} \|x(t_{k+1}) - x^m(t_k)\| + \sum_{k=0}^{n-1} \|x^m(t_k) - x(t_k)\| + \|x^m\|_{1-var,[0,T]} \\ &\le 1 + \sup_n \|x^n\|_{1-var,[0,T]}.\end{aligned}$$

Thus, we have

$$\|x\|_{1-var,[0,T]} \le 1 + \sup_n \|x^n\|_{1-var,[0,T]} < \infty.$$

□

For approximations purposes, it is important to observe that the set of smooth paths is not dense in $C^{1-var}([0,T], \mathbb{R}^d)$ for the 1-variation convergence topology. The closure of the set of smooth paths in the 1-variation norm, which shall be denoted by $C^{0,1-var}([0,T], \mathbb{R}^d)$ is the set of absolutely continuous paths.

Proposition 1.6. *Let $x \in C^{1-var}([0,T], \mathbb{R}^d)$. Then, $x \in C^{0,1-var}([0,T], \mathbb{R}^d)$ if and only if there exists $y \in L^1([0,T])$ such that,*

$$x(t) = x(0) + \int_0^t y(s)ds.$$

PROOF. First let us assume that

$$x(t) = x(0) + \int_0^t y(s)ds,$$

for some $y \in L^1([0,T])$. Since smooth paths are dense in $L^1([0,T])$, we can find a sequence y^n in $L^1([0,T])$ such that $\|y - y^n\|_1 \to 0$. Define then,

$$x^n(t) = x(0) + \int_0^t y^n(s)ds.$$

We have

$$\|x - x^n\|_{1-var,[0,T]} = \|y - y^n\|_1.$$

This implies that $x \in C^{0,1-var}([0,T],\mathbb{R}^d)$. Conversely, if $x \in C^{0,1-var}([0,T],\mathbb{R}^d)$, there exists a sequence of smooth paths x^n that converges in the 1-variation topology to x. Each x^n can be written as,

$$x^n(t) = x^n(0) + \int_0^t y^n(s)ds.$$

We still have

$$\|x^m - x^n\|_{1-var,[0,T]} = \|y^m - y^n\|_1,$$

so that y^n converges to some y in L^1. It is then clear that

$$x(t) = x(0) + \int_0^t y(s)ds,$$

□

Exercise 1.7. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. Show that x is the limit in 1-variation of piecewise linear interpolations if and only if $x \in C^{0,1-var}([0,T],\mathbb{R}^d)$.*

2. Riemann-Stieltjes integrals and Gronwall's lemma

Let $y : [0,T] \to \mathbb{R}^{e\times d}$ be a piecewise continuous path and $x \in\in C^{1-var}([0,T],\mathbb{R}^d)$. It is well-known that we can integrate y against x by using the Riemann-Stieltjes integral which is a natural extension of the Riemann integral. The idea is to use the Riemann sums

$$\sum_{k=0}^{n-1} y(t_k)(x(t_{k+1}) - x(t_k)),$$

where $\Pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$. It is easy to prove that, when the mesh of the subdivision Π goes to 0, the Riemann sums converge to a limit which is independent from the sequence of subdivisions that was chosen. The limit is then denoted $\int_0^T y(t)dx(t)$ and called the Riemann-Stieltjes integral of y against x. Since x has a bounded variation, it is easy to see that, more generally,

$$\sum_{k=0}^{n-1} y(\xi_k)(x(t_{k+1}) - x(t_k)),$$

with $t_k \le \xi_k \le t_{k+1}$ would also converge to $\int_0^T y(t)dx(t)$. If

$$x(t) = x(0) + \int_0^t g(s)ds$$

is an absolutely continuous path, then it is not difficult to prove that we have

$$\int_0^T y(t)dx(t) = \int_0^T y(t)g(t)dt,$$

where the integral on the right hand side is understood in Riemann's sense.

We have

$$\begin{aligned}\left\|\sum_{k=0}^{n-1} y(t_k)(x(t_{k+1})-x(t_k))\right\| &\le \sum_{k=0}^{n-1} \|y(t_k)\|\|(x(t_{k+1})-x(t_k))\| \\ &\le \sum_{k=0}^{n-1} \|y(t_k)\|\|(x(t_{k+1})-x(t_k))\| \\ &\le \sum_{k=0}^{n-1} \|y(t_k)\|\|x\|_{1-var,[t_k,t_{k+1}]}.\end{aligned}$$

Thus, by taking the limit when the mesh of the subdivision goes to 0, we obtain the estimate

$$\left\|\int_0^T y(t)dx(t)\right\| \le \int_0^T \|y(t)\|\|dx(t)\| \le \|y\|_{\infty,[0,T]}\|x\|_{1-var,[0,T]},$$

where $\int_0^T \|y(t)\|\|dx(t)\|$ is the notation for the Riemann-Stieltjes integral of $\|y\|$ against the bounded variation path $l(t) = \|x\|_{1-var,[0,t]}$. We can also estimate the Riemann-Stieltjes integral in the 1-variation distance. We collect the following estimate for later use

Proposition 2.1. *Let $y, y' : [0,T] \to \mathbb{R}^{e\times d}$ be a piecewise continuous path and $x, x' \in C^{1-var}([0,T],\mathbb{R}^d)$. We have*

$$\left\|\int_0^{\cdot} y'(t)dx'(t) - \int_0^{\cdot} y(t)dx(t)\right\|_{1-var,[0,T]} \le \|x\|_{1-var,[0,T]}\|y-y'\|_{\infty,[0,T]}+\|y'\|_{\infty,[0,T]}\|x-x'\|_{1-var,[0,T]}.$$

The Riemann-Stieltjes satisfies the usual rules of calculus, for instance the integration by parts formula takes the following form

Proposition 2.2. *Let $y \in C^{1-var}([0,T],\mathbb{R}^{e\times d})$ and $x \in C^{1-var}([0,T],\mathbb{R}^d)$.*

$$\int_0^T y(t)dx(t) + \int_0^T dy(t)x(t) = y(T)x(T) - y(0)x(0).$$

We also have the following change of variable formula:

Proposition 2.3. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $\Phi : \mathbb{R}^d \to \mathbb{R}^e$ be a C^1 map. We have*

$$\Phi(x(T)) = \Phi(x(0)) + \int_0^T \Phi'(x(t))dx(t).$$

Proof. From the mean value theorem

$$\Phi(x(T)) - \Phi(x(0)) = \sum_{k=0}^{n-1}(\Phi(x(t_{k+1})) - \Phi(x(t_k))) = \sum_{k=0}^{n-1}\Phi'(x_{\xi_k})(x(t_{k+1})-x(t_k)),$$

with $t_k \le \xi_k \le t_{k+1}$. The result is then obtained by taking the limit when the mesh of the subdivision goes to 0. □

We finally state a classical analysis lemma, Gronwall's lemma, which provides a wonderful tool to estimate solutions of differential equations.

Proposition 2.4. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $\Phi : [0,T] \to [0,\infty)$ be a bounded measurable function. If,*

$$\Phi(t) \le A + B\int_0^t \Phi(s)\|dx(s)\|, \quad 0 \le t \le T,$$

for some $A, B \ge 0$, then

$$\Phi(t) \le A\exp(B\|x\|_{1-var,[0,t]}) \quad 0 \le t \le T.$$

Proof. Iterating the inequality

$$\Phi(t) \le A + B\int_0^t \Phi(s)\|dx(s)\|$$

N times, we get

$$\Phi(t) \le A + \sum_{k=1}^{n} AB^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \|dx(t_k)\| \cdots \|dx(t_1)\| + R_n(t),$$

where $R_n(t)$ is a remainder term that goes to 0 when $n \to \infty$. Observing that

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \|dx(t_k)\| \cdots \|dx(t_1)\| = \frac{\|x\|^k_{1-var,[0,t]}}{k!}$$

and sending n to ∞ finishes the proof. □

3. Differential equations driven by bounded variation paths

We now turn to the basic existence and uniqueness results concerning differential equations driven by bounded variation paths.

Theorem 3.1. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a Lipschitz continuous map, that is there exists a constant $K > 0$ such that for every $x, y \in \mathbb{R}^d$,*

$$\|V(x) - V(y)\| \le K\|x - y\|.$$

For every $y_0 \in \mathbb{R}^e$, there is a unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

Moreover $y \in C^{1-var}([0,T],\mathbb{R}^e)$.

Proof. The proof is a classical application of the fixed point theorem. Let $0 < \tau \le T$ and consider the map Φ going from the space of continuous functions $[0,\tau] \to \mathbb{R}^e$ into itself, which is defined by

$$\Phi(y)_t = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le \tau.$$

By using estimates on Riemann-Stieltjes integrals, we deduce that

$$\begin{aligned}\|\Phi(y^1) - \Phi(y^2)\|_{\infty,[0,\tau]} &\le \|V(y^1) - V(y^2)\|_{\infty,[0,\tau]}\|x\|_{1-var,[0,\tau]} \\ &\le K\|y^1 - y^2\|_{\infty,[0,\tau]}\|x\|_{1-var,[0,\tau]}\end{aligned}$$

If τ is small enough, then $K\|x\|_{1-var,[0,\tau]} < 1$, which means that Φ is a contraction that admits a unique fixed point y. This y is the unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le \tau.$$

By considering then a subdivision

$$\{\tau = \tau_1 < \tau_2 < \cdots < \tau_n = T\}$$

such that $K\|x\|_{1-var,[\tau_k,\tau_{k+1}]} < 1$, we obtain a unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

□

The solution of a differential equation is a continuous function of the initial condition, more precisely we have the following estimate:

Proposition 3.2. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a Lipschitz continuous map such that for every $x, y \in \mathbb{R}^d$,*

$$\|V(x) - V(y)\| \le K\|x - y\|.$$

If y^1 and y^2 are the solutions of the differential equations:

$$y^1(t) = y^1(0) + \int_0^t V(y^1(s))dx(s), \quad 0 \le t \le T,$$

and

$$y^2(t) = y^2(0) + \int_0^t V(y^2(s))dx(s), \quad 0 \le t \le T,$$

then the following estimate holds:

$$\|y^1 - y^2\|_{\infty,[0,T]} \le \|y^1(0) - y^2(0)\| \exp\left(K\|x\|_{1-var,[0,T]}\right).$$

Proof. We have

$$\|y^1 - y^2\|_{\infty,[0,t]} \le \|y^1(0) - y^2(0)\| + K\int_0^t \|y^1 - y^2\|_{\infty,[0,s]}\|dx(s)\|,$$

and conclude by Gronwall's lemma. □

This continuity can be understood in terms of flows. Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a Lipschitz map. Denote by $\pi(t, y_0)$, $0 \le t \le T$, $y_0 \in \mathbb{R}^e$, the unique solution of the equation

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

The previous proposition shows that for a fixed $0 \le t \le T$, the map $y_0 \to \pi(t, y_0)$ is Lipschitz continuous. The set $\{\pi(t,\cdot), 0 \le t \le T\}$ is called the flow of the equation. Under more regularity assumptions on V, the $y_0 \to \pi(t, y_0)$ is even C^1 and the Jacobian map solves a linear equation.

Proposition 3.3. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a C^1 Lipschitz continuous map. Let $\pi(t,y_0)$ be the flow of the equation*

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

Then for every $0 \le t \le T$, the map $y_0 \to \pi(t,y_0)$ is C^1 and the Jacobian $J_t = \frac{\partial \pi(t,y_0)}{\partial y_0}$ is the unique solution of the matrix linear equation

$$J_t = Id + \int_0^t DV(\pi(s,y_0))J_s dx(s).$$

We finally turn to the important estimate showing that solutions of differential equations are continuous with respect to the driving path in the 1-variation topology

Theorem 3.4. *Let $x^1, x^2 \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^d \to \mathbb{R}^e$ be a Lipschitz and bounded continuous map such that for every $x, y \in \mathbb{R}^d$,*

$$\|V(x) - V(y)\| \le K\|x - y\|.$$

If y^1 and y^2 are the solutions of the differential equations:

$$y^1(t) = y(0) + \int_0^t V(y^1(s))dx^1(s), \quad 0 \le t \le T,$$

and

$$y^2(t) = y(0) + \int_0^t V(y^2(s))dx^2(s), \quad 0 \le t \le T,$$

then the following estimate holds:

$$\|y^1 - y^2\|_{1-var,[0,T]} \le \|V\|_\infty \left(1 + K\|x\|_{1-var,[0,T]} \exp\left(K\|x\|_{1-var,[0,T]}\right)\right) \|x^1 - x^2\|_{1-var,[0,T]}.$$

Proof. We first give an estimate in the supremum topology. It is easily seen that the assumptions imply

$$\|y^1 - y^2\|_{\infty,[0,t]} \le K \int_0^t \|y^1 - y^2\|_{\infty,[0,s]} \|dx^1(s)\| + \|V\|_\infty \|x^1 - x^2\|_{1-var,[0,T]}.$$

From Gronwall's lemma, we deduce that

$$\|y^1 - y^2\|_{\infty,[0,T]} \le \|V\|_\infty \exp\left(K\|x\|_{1-var,[0,T]}\right) \|x^1 - x^2\|_{1-var,[0,T]}.$$

Now, we also have for any $0 \le s \le t \le T$,

$$\|y^1(t) - y^2(t) - (y^1(s) - y^2(s))\| \le K\|y^1 - y^2\|_{\infty,[0,T]}\|x^1\|_{1-var,[s,t]} + \|V\|_\infty\|x^1 - x^2\|_{1-var,[s,t]}.$$

This implies,

$$\|y^1 - y^2\|_{1-var,[0,T]} \le K\|y^1 - y^2\|_{\infty,[0,T]}\|x^1\|_{1-var,[0,T]} + \|V\|_\infty\|x^1 - x^2\|_{1-var,[0,T]}$$

and yields the conclusion. □

4. Exponential of vector fields and solutions of differential equations

Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^e \to \mathbb{R}^{e\times d}$ be a Lipschitz continuous map. In order to analyse the solution of the differential equation,

$$y(t) = y_0 + \int_0^t V(y(s))dx(s),$$

and make the geometry enter into the scene, it is convenient to see V as a collection of vector fields $V = (V_1, \cdots, V_d)$, where the V_i's are the columns of the matrix V. The differential equation then of course writes

$$y(t) = y_0 + \sum_{i=1}^d \int_0^t V_i(y(s))dx^i(s),$$

Generally speaking, a vector field V on $\mathbb{R}^e$ is a map

$$\begin{array}{cccc} V: & \mathbb{R}^e & \to & \mathbb{R}^e \\ & x & \to & (v_1(x), ..., v_n(x)). \end{array}$$

A vector field V can be seen as a differential operator acting on differentiable functions $f : \mathbb{R}^e \to \mathbb{R}$ as follows:

$$Vf(x) = \langle V(x), \nabla f(x)\rangle = \sum_{i=1}^n v_i(x)\frac{\partial f}{\partial x_i}.$$

We note that V is a derivation, that is for $f, g \in \mathcal{C}^1(\mathbb{R}^e, \mathbb{R})$,

$$V(fg) = (Vf)g + f(Vg).$$

For this reason we often use the differential notation for vector fields and write:

$$V = \sum_{i=1}^d v_i(x)\frac{\partial}{\partial x_i}.$$

Using this action of vector fields on functions, the change of variable formula for solutions of differential equations takes a particularly concise form:

Proposition 4.1. *Let y be a solution of a differential equation that writes*

$$y(t) = y_0 + \sum_{i=1}^d \int_0^t V_i(y(s))dx^i(s),$$

then for any C^1 function $f : \mathbb{R}^e \to \mathbb{R}$,

$$f(y(t)) = f(y_0) + \sum_{i=1}^d \int_0^t V_i f(y(s))dx^i(s),$$

Let V be a Lipschitz vector field on $\mathbb{R}^e$. For any $y_0 \in \mathbb{R}^e$, the differential equation

$$y(t) = y_0 + \int_0^t V(y(s))ds$$

has a unique solution $y : \mathbb{R} \to \mathbb{R}^e$. By time homogeneity of the equation, the flow of this equation satisfies

$$\pi(t_1, \pi(t_2, y_0)) = \pi(t_1 + t_2, y_0).$$

and therefore $\{\pi(t, \cdot), t \in \mathbb{R}\}$ is a one parameter group of diffeomorphisms $\mathbb{R}^e \to \mathbb{R}^e$. This group is generated by V in the sense that for every $y_0 \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{\pi(t, y_0) - y_0}{t} = V(y_0).$$

For these reasons, we write $\pi(t, y_0) = e^{tV}(y_0)$.

Let us now assume that V is a C^1 Lipschitz vector field on $\mathbb{R}^e$. If $\phi : \mathbb{R}^e \to \mathbb{R}^e$ is a diffeomorphism, the pull-back $\phi^* V$ of the vector field V by the map ϕ is the vector field defined by the chain rule,

$$\phi^* V(x) = (d\phi^{-1})_{\phi(x)} \left(V(\phi(x)) \right), \ x \in \mathcal{O}'.$$

In particular, if V' is another C^1 Lipschitz vector field on $\mathbb{R}^e$, then for every $t \in \mathbb{R}$, we have a vector field $(e^{tV})^* V'$. The Lie bracket $[V, V']$ between V and V' is then defined as

$$[V, V'] = \left(\frac{d}{dt} \right)_{t=0} (e^{tV})^* V'.$$

It is computed that

$$[V, V'](x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} v_j(x) \frac{\partial v_i'}{\partial x_j}(x) - v_j'(x) \frac{\partial v_i}{\partial x_j}(x) \right) \frac{\partial}{\partial x_i}.$$

Observe that the Lie bracket obviously satisfies $[V, V'] = -[V', V]$ and the so-called Jacobi identity that is:

$$[V, [V', V'']] + [V', [V'', V]] + [V'', [V, V']] = 0.$$

What the Lie bracket $[V, V']$ really quantifies is the lack of commutativity of the respective flows generated by V and V'.

Lemma 4.2. *Let V, V' be two C^1 Lipschitz vector fields on $\mathbb{R}^e$. Then, $[V, V'] = 0$ if and only if for every $s, t \in \mathbb{R}$,*

$$e^{sV} e^{tV'} = e^{sV + tV'} = e^{tV'} e^{sV}.$$

Proof. This is a classical result in differential geometry, so we only give one part the proof. From the very definition of the Lie bracket and the multiplicativity of the flow, we see that $[V, V'] = 0$ if and only if for every $s \in \mathbb{R}$, $(e^{sV})^* V' = V'$. Now, suppose that $[V, V'] = 0$. Let y be the solution of the equation

$$y(t) = y_0 + \int_0^t V'(y(s)) ds.$$

Since $(e^{sV})^* V' = V'$, we obtain that $e^{sV}(y(t))$ is also a solution of the equation. By uniqueness of solutions, we obtain that

$$e^{sV}(y(t)) = e^{tV'}(e^{sV}(y_0)).$$

As a conclusion,

$$e^{sV}e^{tV'} = e^{tV'}e^{sV}.$$

□

If we consider a differential equation

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s),$$

as we will see it throughout this class, the Lie brackets $[V_i, V_j]$ play an important role in understanding the geometry of the set of solutions. The easiest result in that direction is the following:

Theorem 4.3. *Let $x \in C^{1-var}([0,T], \mathbb{R}^d)$ and let $V_1, \cdots, V_d$ be C^1 Lipschitz vector fields on $\mathbb{R}^e$. Assume that for every $1 \le i, j \le d$, $[V_i, V_j] = 0$, then the solution of the differential equation*

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s), \quad 0 \le t \le T,$$

can be represented as

$$y(t) = \exp\left(\sum_{i=1}^{d} x^i(t)V_i\right)(y_0).$$

Proof. Let

$$F(x_1, \cdots, x_n) = \exp\left(\sum_{i=1}^{d} x_i V_i\right)(y_0).$$

Since the flows generated by the V_i's are commuting, we get that

$$\frac{\partial F}{\partial x_i}(x) = V_i(F(x)).$$

The change of variable formula for bounded variation paths implies then that $F(x^1(t), \cdots, x^n(t))$ is a solution and we conclude by uniqueness. □

CHAPTER 3

Young's integrals

1. p-variation paths

Our next goal in this course is to define an integral that can be used to integrate rougher paths than bounded variation. As we are going to see, Young's integration theory allows to define $\int ydx$ as soon as y has finite q-variation and x and has a finite p-variation with $1/p + 1/q > 1$. This integral is simply is a limit of Riemann sums as for the Riemann-Stiletjes integral. In this lecture we present some basic properties of the space of continuous paths with a finite p-variation. We present these results for $\mathbb{R}^d$ valued paths but most of the results extend without difficulties to paths valued in metric spaces (see chapter 5 in the book by Friz-Victoir).

Definition 1.1. *A path $x : [s,t] \to \mathbb{R}^d$ is said to be of finite p-variation, $p > 0$ the p-variation of x on $[s,t]$, which is defined as*

$$\|x\|_{p-var;[s,t]} := \left(\sup_{\Pi \in \Delta[s,t]} \sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^p \right)^{1/p},$$

is finite. The space of continuous paths $x : [s,t] \to \mathbb{R}^d$ with a finite p-variation will be denoted by $C^{p-var}([s,t], \mathbb{R}^d)$.

The notion of p-variation is only interesting when $p \geq 1$.

Proposition 1.2. *Let $x : [s,t] \to \mathbb{R}^d$ be a continuous path of finite p-variation with $p < 1$. Then, x is constant.*

Proof. We have for $s \leq u \leq t$,

$$\begin{aligned}\|x(u) - x(s)\| &\leq (\max \|x(t_{k+1}) - x(t_k)\|^{1-p}) \left(\sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^p \right) \\ &\leq (\max \|x(t_{k+1}) - x(t_k)\|^{1-p}) \|x\|^p_{p-var;[s,t]}.\end{aligned}$$

Since x is continuous, it is also uniformly continuous on $[s,t]$. By taking a sequence of subdivisions whose mesh tends to 0, we deduce then that

$$\|x(u) - x(s)\| = 0,$$

so that x is constant. □

The following proposition is immediate:

Proposition 1.3. *Let $x : [s,t] \to \mathbb{R}^d$, be a continuous path. If $p \le p'$ then*

$$\|x\|_{p'-var;[s,t]} \le \|x\|_{p-var;[s,t]}.$$

As a consequence

$$C^{p-var}([s,t],\mathbb{R}^d) \subset C^{p'-var}([s,t],\mathbb{R}^d)$$

We remind that a continuous map $\omega : \{0 \le s \le t \le T\} \to [0,\infty)$ that vanishes on the diagonal is called a control f if for all $s \le t \le u$,

$$\omega(s,t) + \omega(t,u) \le \omega(s,u).$$

Proposition 1.4. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$. Then $\omega(s,t) = \|x\|^p_{p-var;[s,t]}$ is a control such that for every $s \le t$,*

$$\|x(s) - x(t)\| \le \omega(s,t)^{1/p}.$$

Proof. It is immediate that

$$\|x(s) - x(t)\| \le \omega(s,t)^{1/p},$$

so we focus on the proof that ω is a control. If $\Pi_1 \in \Delta[s,t]$ and $\Pi_2 \in \Delta[t,u]$, then $\Pi_1 \cup \Pi_2 \in \Delta[s,u]$. As a consequence, we obtain

$$\sup_{\Pi_1\in\Delta[s,t]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\|^p + \sup_{\Pi_2\in\Delta[t,u]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\|^p \le \sup_{\Pi\in\Delta[s,u]} \sum_{k=0}^{n-1} \|x(t_{k+1})-x(t_k)\|^p,$$

thus

$$\|x\|^p_{p-var,[s,t]} + \|x\|^p_{p-var,[t,u]} \le \|x\|^p_{p-var,[s,u]}.$$

The proof of the continuity is left to the reader (see also Proposition 5.8 in the book by Friz-Victoir). □

In the following sense, $\|x\|^p_{p-var;[s,t]}$ is the minimal control of a path x.

Proposition 1.5. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and let $\omega : \{0 \le s \le t \le T\} \to [0,\infty)$ be a control such that for $0 \le s \le t \le T$,*

$$\|x(s) - x(t)\| \le C\omega(s,t)^{1/p},$$

then

$$\|x\|_{p-var;[s,t]} \le C\omega(s,t).$$

Proof. We have

$$\begin{aligned}
\|x\|_{p-var;[s,t]} &= \left(\sup_{\Pi\in\Delta[s,t]} \sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^p \right)^{1/p} \\
&\le \left(\sup_{\Pi\in\Delta[s,t]} \sum_{k=0}^{n-1} C^p \omega(t_k, t_{k+1}) \right)^{1/p} \\
&\le C\omega(s,t).
\end{aligned}$$

□

The next result shows that the set of continuous paths with bounded p-variation is a Banach space.

Theorem 1.6. *Let $p \geq 1$. The space $C^{p-var}([0,T],\mathbb{R}^d)$ endowed with the norm $\|x(0)\| + \|x\|_{p-var,[0,T]}$ is a Banach space.*

Proof. The proof is identical to the case $p = 1$, so we let the careful reader check the details. □

Again, the set of smooth paths is not dense in $C^{p-var}([0,T],\mathbb{R}^d)$ for the p-variation convergence topology. The closure of the set of smooth paths in the p-variation norm shall be denoted by $C^{0,p-var}([0,T],\mathbb{R}^d)$. We have the following characterization of paths in $C^{0,p-var}([0,T],\mathbb{R}^d)$.

Proposition 1.7. *Let $p \geq 1$. $x \in C^{0,p-var}([0,T],\mathbb{R}^d)$ if and only if*

$$\lim_{\delta \to 0} \sup_{\Pi \in \Delta[s,t], |\Pi| \leq \delta} \sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^p = 0.$$

Proof. See Theorem 5.31 in the book by Friz-Victoir. □

The following corollary shall often be used in the sequel:

Corollary 1.8. *If $1 \leq p < q$, then $C^{p-var}([0,T],\mathbb{R}^d) \subset C^{0,q-var}([0,T],\mathbb{R}^d)$.*

Proof. Let $\Pi \in \Delta[s,t]$ whose mesh is less than $\delta > 0$. We have

$$\begin{aligned}\sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^q &\leq \left(\sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^p\right) \max \|x(t_{k+1}) - x(t_k)\|^{p-q} \\ &\leq \|x\|^p_{p-var;[s,t]} \max \|x(t_{k+1}) - x(t_k)\|^{p-q}.\end{aligned}$$

As a consequence, we obtain

$$\lim_{\delta \to 0} \sup_{\Pi \in \Delta[s,t], |\Pi| \leq \delta} \sum_{k=0}^{n-1} \|x(t_{k+1}) - x(t_k)\|^q = 0.$$

□

2. Young's integrals

In this lecture we define the Young's integral $\int y dx$ when $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{q-var}([0,T],\mathbb{R}^{e\times d})$ with $\frac{1}{p} + \frac{1}{q} > 1$. The cornerstone is the following Young-Lóeve estimate.

Theorem 2.1. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$ and $y \in C^{1-var}([0,T],\mathbb{R}^{e\times d})$. Consider now $p, q \geq 1$ with $\theta = \frac{1}{p} + \frac{1}{q} > 1$. The following estimate holds: for $0 \leq s \leq t \leq T$,*

$$\left\| \int_s^t y(u)dx(u) - y(s)(x(t) - x(s)) \right\| \leq \frac{1}{1 - 2^{1-\theta}} \|x\|_{p-var;[s,t]} \|y\|_{q-var;[s,t]}.$$

PROOF. For $0 \le s \le t \le T$, let us define

$$\Gamma_{s,t} = \int_s^t y(u)dx(u) - y(s)(x(t) - x(s)).$$

We have for $s < t < u$,

$$\begin{aligned}\Gamma_{s,u} - \Gamma_{s,t} - \Gamma_{t,u} &= -y(s)(x(u) - x(s)) + y(s)(x(t) - x(s)) + y(t)(x(u) - x(t)) \\ &= (y(s) - y(t))(x(t) - x(u)).\end{aligned}$$

As a consequence, we get

$$\|\Gamma_{s,u}\| \le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \|x\|_{p-var;[t,u]}\|y\|_{q-var;[s,t]}.$$

Let now $\omega(s,t) = \|x\|^{1/\theta}_{p-var;[s,t]}\|y\|^{1/\theta}_{q-var;[s,t]}$. We claim that ω is a control. The continuity and the vanishing on the diagonal are obvious to check, so we just need to justify the superadditivity. Let $s < t < u$, we have from Hölder's inequality,

$$\begin{aligned}\omega(s,t) + \omega(t,u) &= \|x\|^{1/\theta}_{p-var;[s,t]}\|y\|^{1/\theta}_{q-var;[s,t]} + \|x\|^{1/\theta}_{p-var;[t,u]}\|y\|^{1/\theta}_{q-var;[t,u]} \\ &\le (\|x\|^{p}_{p-var;[s,t]} + \|x\|^{p}_{p-var;[t,u]})^{\frac{1}{p\theta}}(\|y\|^{q}_{q-var;[s,t]} + \|y\|^{q}_{q-var;[t,u]})^{\frac{1}{q\theta}} \\ &\le \|x\|^{1/\theta}_{p-var;[s,u]}\|y\|^{1/\theta}_{q-var;[s,u]} = \omega(s,u).\end{aligned}$$

We have then

$$\|\Gamma_{s,u}\| \le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \omega(s,u)^{\theta}.$$

For $\varepsilon > 0$, consider then the control

$$\omega_\varepsilon(s,t) = \omega(s,t) + \varepsilon(\|x\|_{1-var;[s,t]} + \|y\|_{1-var;[s,t]})$$

Define now

$$\Psi(r) = \sup_{s,u,\omega_\varepsilon(s,u)\le r} \|\Gamma_{s,u}\|.$$

If s,u is such that $\omega_\varepsilon(s,u) \le r$, we can find a t such that $\omega_\varepsilon(s,t) \le \frac{1}{2}\omega_\varepsilon(s,u)$, $\omega_\varepsilon(t,u) \le \frac{1}{2}\omega_\varepsilon(s,u)$. Indeed, the continuity of ω_ε forces the existence of a t such that $\omega_\varepsilon(s,t) = \omega_\varepsilon(t,u)$. We obtain therefore

$$\|\Gamma_{s,u}\| \le 2\Psi(r/2) + r^{\theta},$$

which implies by maximization,

$$\Psi(r) \le 2\Psi(r/2) + r^{\theta}.$$

By iterating n times this inequality, we obtain

$$\begin{aligned}\Psi(r) &\le 2^n\Psi\left(\frac{r}{2^n}\right) + \sum_{k=0}^{n-1} 2^{k(1-\theta)}r^{\theta} \\ &\le 2^n\Psi\left(\frac{r}{2^n}\right) + \frac{1}{1-2^{1-\theta}}r^{\theta}.\end{aligned}$$

It is now clear that

$$\begin{aligned}\|\Gamma_{s,t}\| &\le \left\| \int_s^t (y(u)-y(s))dx(u) \right\| \\ &\le \|x\|_{1-var;[s,t]} \|y-y(s)\|_{\infty;[s,t]} \\ &\le (\|x\|_{1-var;[s,t]} + \|y\|_{1-var;[s,t]})^2 \\ &\le \frac{1}{\varepsilon^2}\omega_\varepsilon(s,t)^2,\end{aligned}$$

so that

$$\lim_{n\to\infty} 2^n \Psi\left(\frac{r}{2^n}\right) = 0.$$

We conclude

$$\Psi(r) \le \frac{1}{1-2^{1-\theta}} r^\theta$$

and thus

$$\|\Gamma_{s,u}\| \le \frac{1}{1-2^{1-\theta}} \omega_\varepsilon(s,u)^\theta.$$

Sending $\varepsilon \to 0$, finishes the proof. □

It is remarkable that the Young-Lóeve estimate only involves $\|x\|_{p-var;[s,t]}$ and $\|y\|_{q-var;[s,t]}$. As a consequence, we obtain the following result whose proof is let to the reader:

Proposition 2.2. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{q-var}([0,T],\mathbb{R}^{e\times d})$ with $\theta = \frac{1}{p} + \frac{1}{q} > 1$. Let us assume that there exists a sequence $x^n \in C^{1-var}([0,T],\mathbb{R}^d)$ such that $x^n \to x$ in $C^{p-var}([0,T],\mathbb{R}^d)$ and a sequence $y^n \in C^{1-var}([0,T],\mathbb{R}^{e\times d})$ such that $y^n \to x$ in $C^{q-var}([0,T],\mathbb{R}^d)$, then for every $s < t$, $\int_s^t y^n(u)dx^n(u)$ converges to a limit that we call the Young's integral of y against x on the interval $[s,t]$ and denote $\int_s^t y(u)dx(u)$. The integral $\int_s^t y(u)dx(u)$ does not depend of the sequences x^n and y^n and the following estimate holds: for $0 \le s \le t \le T$,*

$$\left\| \int_s^t y(u)dx(u) - y(s)(x(t)-x(s)) \right\| \le \frac{1}{1-2^{1-\theta}} \|x\|_{p-var;[s,t]} \|y\|_{q-var;[s,t]}.$$

The closure of $C^{1-var}([0,T],\mathbb{R}^d)$ in $C^{p-var}([0,T],\mathbb{R}^d)$ is $C^{0,p-var}([0,T],\mathbb{R}^d)$ and we know that $C^{p+\varepsilon-var}([0,T],\mathbb{R}^d) \subset C^{0,p-var}([0,T],\mathbb{R}^d)$. It is therefore obvious to extend the Young's integral for every $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{q-var}([0,T],\mathbb{R}^{e\times d})$ with $\theta = \frac{1}{p} + \frac{1}{q} > 1$ and the Young-Lóeve estimate still holds

$$\left\| \int_s^t y(u)dx(u) - y(s)(x(t)-x(s)) \right\| \le \frac{1}{1-2^{1-\theta}} \|x\|_{p-var;[s,t]} \|y\|_{q-var;[s,t]}.$$

From this estimate, we easily see that for $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{p-var}([0,T],\mathbb{R}^{e\times d})$ with $\frac{1}{p} + \frac{1}{q} > 1$ the sequence of Riemann sums

$$\sum_{k=0}^{n-1} y(t_i)(x_{t_{i+1}} - x_{t_i})$$

will converge to $\int_s^t y(u)dx(u)$ when the mesh of the subdivision goes to 0. We record for later use the following estimate on the Young's integral, which is also an easy consequence of the Young-Lóeve estimate (see Theorem 6.8 in the book for further details).

Proposition 2.3. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{q-var}([0,T],\mathbb{R}^{e\times d})$ with $\frac{1}{p}+\frac{1}{q}>1$. The integral path $t \to \int_0^t y(u)dx(u)$ is continuous with a finite p-variation and we have*

$$\left\| \int_0^{\cdot} y(u)dx(u) \right\|_{p-var,[s,t]} \le C\|x\|_{p-var;[s,t]} \left(\|y\|_{q-var;[s,t]} + \|y\|_{\infty;[s,t]} \right)$$
$$\le 2C\|x\|_{p-var;[s,t]} \left(\|y\|_{q-var;[s,t]} + \|y(0)\| \right)$$

3. Young's differential equations

In the previous lecture we defined the Young's integral $\int ydx$ when $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and $y \in C^{q-var}([0,T],\mathbb{R}^{e\times d})$ with $\frac{1}{p}+\frac{1}{q}>1$. The integral path $\int_0^t ydx$ has then a bounded p-variation. Now, if $V:\mathbb{R}^e \to \mathbb{R}^{d\times d}$ is a Lipschitz map, then the integral, $\int V(x)dx$ is only defined when $\frac{1}{p}+\frac{1}{p}>1$, that is for $p<2$. With this in mind, it is apparent that Young's integration should be useful to solve differential equations driven by continuous paths with bounded p-variation for $p<2$. If $p \ge 2$, then the Young's integral is of no help and the rough paths theory later explained is the correct one.

The basic existence and uniqueness result is the following. Throughout this lecture, we assume that $p<2$.

Theorem 3.1. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and let $V:\mathbb{R}^e \to \mathbb{R}^{e\times d}$ be a Lipschitz continuous map, that is there exists a constant $K>0$ such that for every $x,y \in \mathbb{R}^e$,*

$$\|V(x)-V(y)\| \le K\|x-y\|.$$

For every $y_0 \in \mathbb{R}^e$, there is a unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

Moreover $y \in C^{p-var}([0,T],\mathbb{R}^e)$.

Proof. The proof is of course based again of the fixed point theorem. Let $0<\tau\le T$ and consider the map Φ going from the space $C^{p-var}([0,\tau],\mathbb{R}^e)$ into itself, which is defined by

$$\Phi(y)_t = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le \tau.$$

By using basic estimates on the Young's integrals, we deduce that

$$\|\Phi(y^1)-\Phi(y^2)\|_{p-var,[0,\tau]} \le C\|x\|_{p-var,[0,\tau]}(\|V(y^1)-V(y^2)\|_{p-var,[0,\tau]} + \|V(y^1)(0)-V(y^2)(0)\|)$$
$$\le CK\|x\|_{p-var,[0,\tau]}(\|y^1-y^2\|_{p-var,[0,\tau]} + \|y^1(0)-y^2(0)\|).$$

If τ is small enough, then $CK\|x\|_{p-var,[0,\tau]}<1$, which means that Φ is a contraction of the Banach space $C^{p-var}([0,\tau],\mathbb{R}^e)$ endowed with the norm $\|y\|_{p-var,[0,\tau]}+\|y(0)\|$.

The fixed point of Φ, let us say y, is the unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le \tau.$$

By considering then a subdivision

$$\{\tau = \tau_1 < \tau_2 < \cdots < \tau_n = T\}$$

such that $CK\|x\|_{p-var,[\tau_k,\tau_{k+1}]} < 1$, we obtain a unique solution to the differential equation:

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

□

As for the bounded variation case, the solution of a Young's differential equation is a C^1 function of the initial condition,

Proposition 3.2. *Let $x \in C^{p-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^e \to \mathbb{R}^{e\times d}$ be a C^1 Lipschitz continuous map. Let $\pi(t, y_0)$ be the flow of the equation*

$$y(t) = y_0 + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

Then for every $0 \le t \le T$, the map $y_0 \to \pi(t, y_0)$ is C^1 and the Jacobian $J_t = \frac{\partial \pi(t,y_0)}{\partial y_0}$ is the unique solution of the matrix linear equation

$$J_t = Id + \sum_{i=1}^d \int_0^t DV_i(\pi(s, y_0))J_s dx^i(s).$$

As we already mentioned it before, solutions of Young's differential equations are continuous with respect to the driving path in the p-variation topology

Theorem 3.3. *Let $x^n \in C^{p-var}([0,T],\mathbb{R}^d)$ and let $V : \mathbb{R}^e \to \mathbb{R}^{e\times d}$ be a Lipschitz and bounded continuous map such that for every $x, y \in \mathbb{R}^d$,*

$$\|V(x) - V(y)\| \le K\|x - y\|.$$

Let y^n be the solution of the differential equation:

$$y^n(t) = y(0) + \int_0^t V(y^n(s))dx^n(s), \quad 0 \le t \le T.$$

If x^n converges to x in p-variation, then y^n converges in p-variation to the solution of the differential equation:

$$y(t) = y(0) + \int_0^t V(y(s))dx(s), \quad 0 \le t \le T.$$

Proof. Let $0 \le s \le t \le T$. We have

$$\begin{aligned}
\|y-y^n\|_{p-var,[s,t]} &= \left\| \int_0^\cdot V(y(u))dx(u) - \int_0^\cdot V(y^n(u))dx^n(u) \right\|_{p-var,[s,t]} \\
&\le \left\| \int_0^\cdot (V(y(u))-V(y^n(u)))dx(u) + \int_0^\cdot V(y^n(u))d(x(u)-x^n(u)) \right\|_{p-var,[s,t]} \\
&\le \left\| \int_0^\cdot (V(y(u))-V(y^n(u)))dx(u) \right\|_{p-var,[s,t]} + \left\| \int_0^\cdot V(y^n(u))d(x(u)-x^n(u)) \right\|_{p-var,[s,t]} \\
&\le CK\|x\|_{p-var,[s,t]}\|y-y^n\|_{p-var,[s,t]} + C\|x-x^n\|_{p-var,[s,t]}(K\|y^n\|_{p-var,[s,t]} + \|V\|_{\infty,[0,T]})
\end{aligned}$$

Thus, if s,t is such that $CK\|x\|_{p-var,[s,t]} < 1$, we obtain

$$\|y-y^n\|_{p-var,[s,t]} \le \frac{C(K\|y^n\|_{p-var,[s,t]} + \|V\|_{\infty,[0,T]})}{1-CK\|x\|_{p-var,[s,t]}}\|x-x^n\|_{p-var,[s,t]}.$$

In the very same way, provided $CK\|x^n\|_{p-var,[s,t]} < 1$, we get

$$\|y^n\|_{p-var,[s,t]} \le \frac{C\|V\|_{\infty,[0,T]}}{1-CK\|x^n\|_{p-var,[s,t]}}.$$

Let us fix $0 < \varepsilon < 1$ and pick a sequence $0 \le \tau_1 \le \cdots \le \tau_m = T$ such that $CK\|x\|_{p-var,[\tau_i,\tau_{i+1}]} + \varepsilon < 1$. Since $\|x^n\|_{p-var,[\tau_i,\tau_{i+1}]} \to \|x\|_{p-var,[\tau_i,\tau_{i+1}]}$, for $n \ge N_1$ with N_1 big enough, we have

$$CK\|x^n\|_{p-var,[\tau_i,\tau_{i+1}]} + \frac{\varepsilon}{2} < 1.$$

We deduce that for $n \ge N_1$,

$$\|y^n\|_{p-var,[\tau_i,\tau_{i+1}]} \le \frac{2}{\varepsilon}C\|V\|_{\infty,[0,T]}$$

and

$$\begin{aligned}
\|y-y^n\|_{p-var,[\tau_i,\tau_{i+1}]} &\le \frac{C(K\frac{2}{\varepsilon}C\|V\|_{\infty,[0,T]} + \|V\|_{\infty,[0,T]})}{1-CK\|x\|_{p-var,[\tau_i,\tau_{i+1}]}}\|x-x^n\|_{p-var,[\tau_i,\tau_{i+1}]} \\
&\le \frac{C}{\varepsilon}\|V\|_{\infty,[0,T]}\left(\frac{2KC}{\varepsilon}+1\right)\|x-x^n\|_{p-var,[\tau_i,\tau_{i+1}]} \\
&\le \frac{C}{\varepsilon}\|V\|_{\infty,[0,T]}\left(\frac{2KC}{\varepsilon}+1\right)\|x-x^n\|_{p-var,[0,T]}.
\end{aligned}$$

For $n \ge N_2$ with $N_2 \ge N_1$ and big enough, we have

$$\|x-x^n\|_{p-var,[0,T]} \le \frac{\varepsilon^3}{m},$$

which implies

$$\|y-y^n\|_{p-var,[0,T]} \le \frac{C}{\varepsilon}\|V\|_{\infty,[0,T]}\left(\frac{2KC}{\varepsilon}+1\right)\varepsilon^3.$$

$\square$

CHAPTER 4

Rough paths

1. The signature of a bounded variation path

In this lecture we introduce the central notion of the signature of a path $x \in C^{1-var}([0,T],\mathbb{R}^d)$ which is a convenient way to encode all the algebraic information on the path x which is relevant to study differential equations driven by x. The motivation for the definition of the seignature comes from formal manipulations on Taylor series.

Let us consider a differential equation

$$y(t) = y(s) + \sum_{i=1}^{d} \int_s^t V_i(y(u))dx^i(u),$$

where the V_i's are smooth vector fields on $\mathbb{R}^n$.

If $f:\mathbb{R}^n \to \mathbb{R}$ is a C^∞ function, by the change of variable formula,

$$f(y(t)) = f(y(s)) + \sum_{i=1}^{d} \int_0^t V_i f(y(u))dx^i(u).$$

Now, a new application of the change of variable formula to $V_i f(y(s))$ leads to

$$f(y(t)) = f(y(s)) + \sum_{i=1}^{d} V_i f(y(s)) \int_s^t dx^i(u) + \sum_{i,j=1}^{d} \int_s^t \int_0^u V_j V_i f(y(v))dx^j(v)dx^i(u).$$

We can continue this procedure to get after N steps

$$f(y(t)) = f(y(s)) + \sum_{k=1}^{N} \sum_{I=(i_1,\cdots,i_k)} (V_{i_1}\cdots V_{i_k} f)(y(s)) \int_{\Delta^k[s,t]} dx^I + R_N(s,t)$$

for some remainder term $R_N(s,t)$, where we used the notations:

(1) $\Delta^k[s,t] = \{(t_1,\cdots,t_k) \in [s,t]^k, s \le t_1 \le t_2 \cdots \le t_k \le t\}$
(2) If $I=(i_1,\cdots,i_k) \in \{1,\cdots,d\}^k$ is a word with length k,

$$\int_{\Delta^k[s,t]} dx^I = \int_{s\le t_1 \le t_2 \le \cdots \le t_k \le t} dx^{i_1}(t_1)\cdots dx^{i_k}(t_k).$$

If we let $N \to +\infty$, assuming $R_N(s,t) \to 0$ (which is by the way true for $t-s$ small enough if the V_i's are analytic), we are led to the formal expansion formula:

$$f(y(t)) = f(y(0)) + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} (V_{i_1}\cdots V_{i_k} f)(y(s)) \int_{\Delta^k[s,t]} dx^I.$$

This shows, at least at the formal level, that all the information given by x on y is contained in the iterated integrals $\int_{\Delta^k[s,t]} dx^I$.

Let $\mathbb{R}[[X_1,...,X_d]]$ be the non commutative algebra over $\mathbb{R}$ of the formal series with d indeterminates, that is the set of series

$$Y = y_0 + \sum_{k=1}^{+\infty} \sum_{I\in\{1,...,d\}^k} a_{i_1,...,i_k} X_{i_1}...X_{i_k}.$$

Definition 1.1. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. The signature of x (or Chen's series) is the formal series:*

$$\mathfrak{S}(x)_{s,t} = 1 + \sum_{k=1}^{+\infty} \sum_{I\in\{1,...,d\}^k} \left(\int_{\Delta^k[s,t]} dx^I\right) X_{i_1}\cdots X_{i_k}, \quad 0 \le s \le t \le T.$$

As we are going to see in the next few lectures, the signature is a fascinating algebraic object. At the source of the numerous properties of the signature lie the following so-called Chen's relations

Lemma 1.2 (Chen's relations)**.** *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. For any word $(i_1,...,i_n) \in \{1,...,d\}^n$ and any $0 \le s \le t \le u \le T$,*

$$\int_{\Delta^n[s,u]} dx^{(i_1,...,i_n)} = \sum_{k=0}^{n} \int_{\Delta^k[s,t]} dx^{(i_1,...,i_k)} \int_{\Delta^{n-k}[t,u]} dx^{(i_{k+1},...,i_n)},$$

where we used the convention that if I is a word with length 0, then $\int_{\Delta^0[0,t]} \circ dx^I = 1$.

Proof. It follows readily by induction on n by noticing that

$$\int_{\Delta^n[s,u]} dx^{(i_1,...,i_n)} = \int_s^u \left(\int_{\Delta^{n-1}[s,t_n]} dx^{(i_1,...,i_{n-1})}\right) dx^{i_n}(t_n).$$

□

To avoid heavy notations, it will be convenient to denote

$$\int_{\Delta^k[s,t]} dx^{\otimes k} = \sum_{I\in\{1,...,d\}^k} \left(\int_{\Delta^k[s,t]} dx^I\right) X_{i_1}\cdots X_{i_k}.$$

This notation actually reflects a natural algebra isomorphism between $\mathbb{R}[[X_1,...,X_d]]$ and $1 \oplus_{k=1}^{+\infty} (\mathbb{R}^d)^{\otimes k}$. With this notation, observe that the signature writes then

$$\mathfrak{S}(x)_{s,t} = 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[s,t]} dx^{\otimes k},$$

and that the Chen's relations become

$$\int_{\Delta^n[s,u]} dx^{\otimes n} = \sum_{k=0}^{n} \int_{\Delta^k[s,t]} dx^{\otimes k} \int_{\Delta^{n-k}[t,u]} dx^{\otimes (n-k)}.$$

The Chen's relations imply the following flow property for the signature:

Lemma 1.3. *Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. For any $0 \le s \le t \le u \le T$,*

$$\mathfrak{S}(x)_{s,u} = \mathfrak{S}(x)_{s,t}\mathfrak{S}(x)_{t,u}$$

Proof. Indeed,

$$\begin{aligned}
\mathfrak{S}(x)_{s,u} &= 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[s,u]} dx^{\otimes k} \\
&= 1 + \sum_{k=1}^{+\infty} \sum_{j=0}^{n} \int_{\Delta^j[s,t]} dx^{\otimes j} \int_{\Delta^{k-j}[t,u]} dx^{\otimes (k-j)} \\
&= \mathfrak{S}(x)_{s,t}\mathfrak{S}(x)_{t,u}.
\end{aligned}$$

□

2. Estimating iterated integrals

In the previous lecture we introduced the signature of a bounded variation path x as the formal series

$$\mathfrak{S}(x)_{s,t} = 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[s,t]} dx^{\otimes k}.$$

If now $x \in C^{p-var}([0,T],\mathbb{R}^d)$, $p \ge 1$ the iterated integrals $\int_{\Delta^k[s,t]} dx^{\otimes k}$ can only be defined as Young integrals when $p < 2$. In this lecture, we are going to define the signature of some (not all) paths with a finite p variation when $p < 2$. The construction is due to Terry Lyons in his seminal paper and this is where the rough paths theory really begins.

For $P \in \mathbb{R}[[X_1, ..., X_d]]$ that can be writen as

$$P = P_0 + \sum_{k=1}^{+\infty} \sum_{I \in \{1,...,d\}^k} a_{i_1,...,i_k} X_{i_1} ... X_{i_k},$$

we define

$$\|P\| = |P_0| + \sum_{k=1}^{+\infty} \sum_{I \in \{1,...,d\}^k} |a_{i_1,...,i_k}| \in [0,\infty].$$

It is quite easy to check that for $P, Q \in \mathbb{R}[[X_1, ..., X_d]]$

$$\|PQ\| \le \|P\|\|Q\|.$$

Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. For $p \ge 1$, we denote

$$\left\| \int dx^{\otimes k} \right\|_{p-var,[s,t]} = \left(\sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} \left\| \int_{\Delta^k[t_i,t_{i+1}]} dx^{\otimes k} \right\|^p \right)^{1/p},$$

where $\mathcal{D}[s,t]$ is the set of subdivisions of the interval $[s,t]$. Observe that for $k \ge 2$, in general

$$\int_{\Delta^k[s,t]} dx^{\otimes k} + \int_{\Delta^k[t,u]} dx^{\otimes k} \ne \int_{\Delta^k[s,u]} dx^{\otimes k}.$$

Actually from the Chen's relations we have

$$\int_{\Delta^n[s,u]} dx^{\otimes n} = \int_{\Delta^n[s,t]} dx^{\otimes k} + \int_{\Delta^n[t,u]} dx^{\otimes k} + \sum_{k=1}^{n-1} \int_{\Delta^k[s,t]} dx^{\otimes k} \int_{\Delta^{n-k}[t,u]} dx^{\otimes (n-k)}.$$

It follows that $\left\| \int dx^{\otimes k} \right\|_{p-var,[s,t]}$ needs not to be the p-variation of $t \to \int_{\Delta^k[s,t]} dx^{\otimes k}$. It is however easy to verify that

$$\left\| \int_{\Delta^k[s,\cdot]} dx^{\otimes k} \right\|_{p-var,[s,t]} \le \left\| \int dx^{\otimes k} \right\|_{p-var,[s,t]}.$$

The first major result of rough paths theory is the following estimate:

Theorem 2.1. *Let $p \ge 1$. There exists a constant $C \ge 0$, depending only on p, such that for every $x \in C^{1-var}([0,T], \mathbb{R}^d)$ and $k \ge 0$,*

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \frac{C^k}{\left(\frac{k}{p}\right)!} \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^k, \quad 0 \le s \le t \le T.$$

By $\left(\frac{k}{p}\right)!$, we of course mean $\Gamma\left(\frac{k}{p}+1\right)$. Some remarks are in order before we prove the result. If $p=1$, then the estimate becomes

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \frac{C^k}{k!} \|x\|^k_{1-var,[s,t]},$$

which is immediately checked because

$$\begin{aligned}
\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| &\le \sum_{I \in \{1,\dots,d\}^k} \left\| \int_{\Delta^k[s,t]} dx^I \right\| \\
&\le \sum_{I \in \{1,\dots,d\}^k} \int_{s \le t_1 \le t_2 \le \dots \le t_k \le t} \|dx^{i_1}(t_1)\| \cdots \|dx^{i_k}(t_k)\| \\
&\le \frac{1}{k!} \|x\|^k_{1-var,[s,t]}.
\end{aligned}$$

We can also observe that for $k \le p$, the estimate is easy to obtain because

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \left\| \int dx^{\otimes k} \right\|_{\frac{p}{k}-var,[s,t]}.$$

So, all the work is to prove the estimate when $k > p$. The proof is split into two lemmas. The first one is a binomial inequality which is actually quite difficult to prove:

Lemma 2.2. *For $x, y > 0$, $n \in \mathbb{N}, n \ge 0$, and $p \ge 1$,*

$$\sum_{j=0}^{n} \frac{x^{j/p}}{\left(\frac{j}{p}\right)!} \frac{y^{(n-j)/p}}{\left(\frac{n-j}{p}\right)!} \le p \frac{(x+y)^{n/p}}{\left(\frac{n}{p}\right)!}.$$

PROOF. See Lemma 2.2.2 in the article by Lyons or this proof for the sharp constant. $\square$

The second one is a lemma that actually already was essentially proved in the Lecture on Young's integral, but which was not explicitly stated.

Lemma 2.3. *Let $\Gamma : \{0 \le s \le t \le T\} \to \mathbb{R}^N$. Let us assume that:*

(1) *There exists a control $\tilde{\omega}$ such that*
$$\lim_{r \to 0} \sup_{(s,t), \tilde{\omega}(s,t) \le r} \frac{\|\Gamma_{s,t}\|}{r} = 0;$$
(2) *There exists a control ω and $\theta > 1, \xi > 0$ such that for $0 \le s \le t \le u \le T$,*
$$\|\Gamma_{s,u}\| \le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \xi \omega(s,u)^\theta.$$

Then, for all $0 \le s < t \le T$,
$$\|\Gamma_{s,t}\| \le \frac{\xi}{1 - 2^{1-\theta}} \omega(s,t)^\theta.$$

PROOF. See the proof of the Young-Loeve estimate or Lemma 6.2 in the book by Friz-Victoir. $\square$

We can now turn to the proof of the main result.

PROOF. Let us denote
$$\omega(s,t) = \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^p.$$

We claim that ω is a control. Indeed for $0 \le s \le t \le u \le T$, we have from Hölder's inequality
$$\begin{aligned}
\omega(s,t) + \omega(t,u) &= \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^p + \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[t,u]} \right)^p \\
&\le \left(\sum_{j=1}^{[p]} \left(\left\| \int dx^{\otimes j} \right\|^{p/j}_{\frac{p}{j}-var,[s,t]} + \left\| \int dx^{\otimes j} \right\|^{p/j}_{\frac{p}{j}-var,[t,u]} \right)^{1/p} \right)^p \\
&\le \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,u]} \right)^p = \omega(s,u).
\end{aligned}$$

It is clear that for some constant $\beta > 0$ which is small enough, we have for $k \le p$,
$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \frac{1}{\beta \left(\frac{k}{p}\right)!} \omega(s,t)^{k/p}.$$

Let us now consider

$$\Gamma_{s,t} = \int_{\Delta^{[p]+1}[s,t]} dx^{\otimes([p]+1)}.$$

From the Chen's relations, for $0 \le s \le t \le u \le T$,

$$\Gamma_{s,u} = \Gamma_{s,t} + \Gamma_{t,u} + \sum_{j=1}^{[p]} \int_{\Delta^j[s,t]} dx^{\otimes j} \int_{\Delta^{[p]+1-j}[t,u]} dx^{\otimes([p]+1-j)}.$$

Therefore,

$$\begin{aligned}
\|\Gamma_{s,u}\| &\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \sum_{j=1}^{[p]} \left\| \int_{\Delta^j[s,t]} dx^{\otimes j} \right\| \left\| \int_{\Delta^{[p]+1-j}[t,u]} dx^{\otimes([p]+1-j)} \right\| \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{1}{\beta^2} \sum_{j=1}^{[p]} \frac{1}{\left(\frac{j}{p}\right)!} \omega(s,t)^{j/p} \frac{1}{\left(\frac{[p]+1-j}{p}\right)!} \omega(t,u)^{([p]+1-j)/p} \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{1}{\beta^2} \sum_{j=0}^{[p]+1} \frac{1}{\left(\frac{j}{p}\right)!} \omega(s,t)^{j/p} \frac{1}{\left(\frac{[p]+1-j}{p}\right)!} \omega(t,u)^{([p]+1-j)/p} \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{1}{\beta^2} p \frac{(\omega(s,t)+\omega(t,u))^{([p]+1)/p}}{\left(\frac{[p]+1}{p}\right)!} \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{1}{\beta^2} p \frac{\omega(s,u)^{([p]+1)/p}}{\left(\frac{[p]+1}{p}\right)!}.
\end{aligned}$$

On the other hand, we have

$$\|\Gamma_{s,t}\| \le A \|x\|^{[p]+1}_{1-var,[s,t]}.$$

We deduce from the previous lemma that

$$\|\Gamma_{s,t}\| \le \frac{1}{\beta^2} \frac{p}{1-2^{1-\theta}} \frac{\omega(s,t)^{([p]+1)/p}}{\left(\frac{[p]+1}{p}\right)!},$$

with $\theta = \frac{[p]+1}{p}$. The general case $k \ge p$ is dealt by induction. The details are let to the reader. □

Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. Since

$$\omega(s,t) = \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^p$$

is a control, the estimate

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \frac{C^k}{\left(\frac{k}{p}\right)!} \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^k, \quad 0 \le s \le t \le T.$$

easily implies that for $k > p$,

$$\left\| \int dx^{\otimes k} \right\|_{1-var,[s,t]} \leq \frac{C^k}{\left(\frac{k}{p}\right)!} \omega(s,t)^{k/p}.$$

We stress that it does not imply a bound on the 1-variation of the path $t \to \int_{\Delta^k[0,t]} dx^{\otimes k}$. What we can get for this path, are bounds in p-variation:

Proposition 2.4. *Let $p \geq 1$. There exists a constant $C \geq 0$, depending only on p, such that for every $x \in C^{1-var}([0,T], \mathbb{R}^d)$ and $k \geq 0$,*

$$\left\| \int_{\Delta^k[0,\cdot]} dx^{\otimes k} \right\|_{p-var,[s,t]} \leq \frac{C^k}{\left(\frac{k}{p}\right)!} \omega(s,t)^{1/p} \omega(0,T)^{\frac{k-1}{p}}$$

where

$$\omega(s,t) = \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^p, \quad 0 \leq s \leq t \leq T.$$

Proof. This is an easy consequence of the Chen's relations. Indeed,

$$\begin{aligned}
\left\| \int_{\Delta^k[0,t]} dx^{\otimes k} - \int_{\Delta^k[0,s]} dx^{\otimes k} \right\| &= \left\| \sum_{j=1}^{k} \int_{\Delta^j[s,t]} dx^{\otimes j} \int_{\Delta^{j-k}[0,s]} dx^{\otimes (k-j)} \right\| \\
&\leq \sum_{j=1}^{k} \left\| \int_{\Delta^j[s,t]} dx^{\otimes j} \right\| \left\| \int_{\Delta^{j-k}[0,s]} dx^{\otimes (k-j)} \right\| \\
&\leq C^k \sum_{j=1}^{k} \frac{1}{\left(\frac{j}{p}\right)!} \omega(s,t)^{j/p} \frac{1}{\left(\frac{k-j}{p}\right)!} \omega(s,t)^{(k-j)/p} \\
&\leq C^k \omega(s,t)^{1/p} \sum_{j=1}^{k} \frac{1}{\left(\frac{j}{p}\right)!} \omega(0,T)^{(j-1)/p} \frac{1}{\left(\frac{k-j}{p}\right)!} \omega(0,T)^{(k-j)/p} \\
&\leq C^k \omega(s,t)^{1/p} \omega(0,T)^{(k-1)/p} \sum_{j=1}^{k} \frac{1}{\left(\frac{j}{p}\right)!} \frac{1}{\left(\frac{k-j}{p}\right)!}.
\end{aligned}$$

and we conclude with the binomial inequality. □

We are now ready for a second major estimate which is the key to define iterated integrals of a path with p-bounded variation when $p \geq 2$.

Theorem 2.5. *Let $p \geq 1$, $K > 0$ and $x, y \in C^{1-var}([0,T], \mathbb{R}^d)$ such that*

$$\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[0,T]} \leq 1,$$

and

$$\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)^p+\left(\sum_{j=1}^{[p]}\left\|\int dy^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)^p\le K.$$

Then there exists a constant $C\ge 0$ depending only on p and K such that for $0\le s\le t\le T$ and $k\ge 1$

$$\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}-\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|\le\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes k}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)\frac{C^k}{\left(\frac{k}{p}\right)!}\omega(s,t)^{k/p},$$

$$\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}\right\|+\left\|\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|\le\frac{C^k}{\left(\frac{k}{p}\right)!}\omega(s,t)^{k/p}$$

where ω is the control

$$\omega(s,t)=\frac{\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[s,t]}\right)^p+\left(\sum_{j=1}^{[p]}\left\|\int dy^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[s,t]}\right)^p}{\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)^p+\left(\sum_{j=1}^{[p]}\left\|\int dy^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)^p}$$
$$+\left(\frac{\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[s,t]}}{\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes j}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}}\right)^p$$

PROOF. We prove by induction on k that for some constants C,β,

$$\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}-\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|\le\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes k}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)\frac{C^k}{\beta\left(\frac{k}{p}\right)!}\omega(s,t)^{k/p},$$

$$\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}\right\|+\left\|\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|\le\frac{C^k}{\beta\left(\frac{k}{p}\right)!}\omega(s,t)^{k/p}$$

For $k\le p$, we trivially have

$$\begin{aligned}\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}-\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|&\le\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes k}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)^k\omega(s,t)^{k/p}\\&\le\left(\sum_{j=1}^{[p]}\left\|\int dx^{\otimes j}-\int dy^{\otimes k}\right\|^{1/j}_{\frac{p}{j}-var,[0,T]}\right)\omega(s,t)^{k/p}.\end{aligned}$$

and

$$\left\|\int_{\Delta^k[s,t]}dx^{\otimes k}\right\|+\left\|\int_{\Delta^k[s,t]}dy^{\otimes k}\right\|\le K^{k/p}\omega(s,t)^{k/p}$$

Not let us assume that the result is true for $0 \le j \le k$ with $k > p$. Let

$$\Gamma_{s,t} = \int_{\Delta^k[s,t]} dx^{\otimes(k+1)} - \int_{\Delta^k[s,t]} dy^{\otimes(k+1)}$$

From the Chen's relations, for $0 \le s \le t \le u \le T$,

$$\Gamma_{s,u} = \Gamma_{s,t}+\Gamma_{t,u}+\sum_{j=1}^{k}\int_{\Delta^j[s,t]} dx^{\otimes j}\int_{\Delta^{k+1-j}[t,u]} dx^{\otimes(k+1-j)}-\sum_{j=1}^{k}\int_{\Delta^j[s,t]} dy^{\otimes j}\int_{\Delta^{k+1-j}[t,u]} dy^{\otimes(k+1-j)}.$$

Therefore, from the binomial inequality

$$\begin{aligned}
\|\Gamma_{s,u}\| \le &\|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \sum_{j=1}^{k} \left\| \int_{\Delta^j[s,t]} dx^{\otimes j} - \int_{\Delta^j[s,t]} dy^{\otimes j} \right\| \left\| \int_{\Delta^{k+1-j}[t,u]} dx^{\otimes(k+1-j)} \right\| \\
&+ \sum_{j=1}^{k} \left\| \int_{\Delta^j[s,t]} dy^{\otimes j} \right\| \left\| \int_{\Delta^{k+1-j}[t,u]} dx^{\otimes(k+1-j)} - \int_{\Delta^{k+1-j}[t,u]} dy^{\otimes(k+1-j)} \right\| \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{1}{\beta^2}\tilde{\omega}(0,T) \sum_{j=1}^{k} \frac{C^j}{\left(\frac{j}{p}\right)!}\omega(s,t)^{j/p} \frac{C^{k+1-j}}{\left(\frac{k+1-j}{p}\right)!}\omega(t,u)^{(k+1-j)/p} \\
&+ \frac{1}{\beta^2}\tilde{\omega}(0,T) \sum_{j=1}^{k} \frac{C^j}{\left(\frac{j}{p}\right)!}\omega(s,t)^{j/p} \frac{C^{k+1-j}}{\left(\frac{k+1-j}{p}\right)!}\omega(t,u)^{(k+1-j)/p} \\
&\le \|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \frac{2p}{\beta^2}\tilde{\omega}(0,T) C^{k+1} \frac{\omega(s,u)^{(k+1)/p}}{\left(\frac{k+1}{p}\right)!}
\end{aligned}$$

where

$$\tilde{\omega}(0,T) = \sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes k} \right\|^{1/j}_{\frac{p}{j}-var,[0,T]}.$$

We deduce

$$\|\Gamma_{s,t}\| \le \frac{2p}{\beta^2(1-2^{1-\theta})}\tilde{\omega}(0,T)C^{k+1}\frac{\omega(s,t)^{(k+1)/p}}{\left(\frac{k+1}{p}\right)!}$$

with $\theta = \frac{k+1}{p}$. A correct choice of β finishes the induction argument. □

In this lecture, it is now time to harvest the fruits of the of the two previous lectures. This will allow us to finally define the notion of p-rough path and to construct the signature of such path.

A first result which is a consequence of the theorem proved in the previous lecture is the following continuity of the iterated iterated integrals with respect to a convenient topology. The proof uses very similar arguments to the previous two lectures, so we let it as an exercise to the student.

Theorem 2.6. *Let $p \ge 1$, $K > 0$ and $x, y \in C^{1-var}([0,T], \mathbb{R}^d)$ such that*

$$\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \le 1,$$

and

$$\left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right)^p + \left(\sum_{j=1}^{[p]} \left\| \int dy^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right)^p \le K.$$

Then there exists a constant $C \ge 0$ depending only on p and K such that for $0 \le s \le t \le T$ and $k \ge 1$

$$\left\| \int_{\Delta^k[0,\cdot]} dx^{\otimes k} - \int_{\Delta^k[0,\cdot]} dy^{\otimes k} \right\|_{p-var,[0,T]} \le \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right) \frac{C^k}{\left(\frac{k}{p}\right)!},$$

This continuity result naturally leads to the following definition.

Definition 2.7. *Let $p \ge 1$ and $x \in C^{p-var}([0,T], \mathbb{R}^d)$. We say that x is a p-rough path if there exists a sequence $x_n \in C^{1-var}([0,T], \mathbb{R}^d)$ such that $x_n \to x$ in p-variation and such that for every $\varepsilon > 0$, there exists $N \ge 0$ such that for $m, n \ge N$,*

$$\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \le \varepsilon.$$

The space of p-rough paths will be denoted $\mathbf{\Omega}^p([0,T], \mathbb{R}^d)$.

From the very definition, $\mathbf{\Omega}^p([0,T], \mathbb{R}^d)$ is the closure of $C^{1-var}([0,T], \mathbb{R}^d)$ inside $C^{p-var}([0,T], \mathbb{R}^d)$ for the distance

$$d_{\mathbf{\Omega}^p([0,T],\mathbb{R}^d)}(x,y) = \sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}.$$

If $x \in \mathbf{\Omega}^p([0,T], \mathbb{R}^d)$ and $x_n \in C^{1-var}([0,T], \mathbb{R}^d)$ such that $x_n \to x$ in p-variation and such that for every $\varepsilon > 0$, there exists $N \ge 0$ such that for $m, n \ge N$,

$$\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \le \varepsilon,$$

then we define $\int_{\Delta^k[s,t]} dx^{\otimes k}$ for $k \le p$ as the limit of the iterated integrals $\int_{\Delta^k[s,t]} dx_n^{\otimes k}$. However it is important to observe that $\int_{\Delta^k[s,t]} dx^{\otimes k}$ may then depend on the choice of the approximating sequence x_n. Once the integrals $\int_{\Delta^k[s,t]} dx^{\otimes k}$ are defined for $k \le p$, we can then use the previous theorem to construct all the iterated integrals $\int_{\Delta^k[s,t]} dx^{\otimes k}$ are defined for $k > p$. It is then obvious that if $x, y \in \mathbf{\Omega}^p([0,T], \mathbb{R}^d)$, then

$$1 + \sum_{k=1}^{[p]} \int_{\Delta^k[s,t]} dx^{\otimes k} = 1 + \sum_{k=1}^{[p]} \int_{\Delta^k[s,t]} dy^{\otimes k}$$

implies that

$$1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[s,t]} dx^{\otimes k} = 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[s,t]} dy^{\otimes k}.$$

In other words the signature of a p-rough path is completely determinated by its truncated signature at order $[p]$:

$$\mathfrak{S}_{[p]}(x)_{s,t} = 1 + \sum_{k=1}^{[p]} \int_{\Delta^k[s,t]} dx^{\otimes k}.$$

For this reason, it is natural to present a p-rough path by this truncated signature at order $[p]$ in order to stress that the choice of the approximating sequence to contruct the iterated integrals up to order $[p]$ has been made. This will be further explained in much more details when we will introduce the notion of geometric rough path over a rough path.

The following results are straightforward to obtain from the previous lectures by a limiting argument.

Lemma 2.8 (Chen's relations)**.** *Let $x \in \mathbf{\Omega}^p([0,T],\mathbb{R}^d)$, $p \geq 1$. For $0 \leq s \leq t \leq u \leq T$, and $n \geq 1$,*

$$\int_{\Delta^n[s,u]} dx^{\otimes n} = \sum_{k=0}^{n} \int_{\Delta^k[s,t]} dx^{\otimes k} \int_{\Delta^{n-k}[t,u]} dx^{\otimes (n-k)}.$$

Theorem 2.9. *Let $p \geq 1$. There exists a constant $C \geq 0$, depending only on p, such that for every $x \in \mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ and $k \geq 1$,*

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \leq \frac{C^k}{\left(\frac{k}{p}\right)!} \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|_{\frac{p}{j}-var,[s,t]}^{1/j} \right)^k, \quad 0 \leq s \leq t \leq T.$$

If $p \geq 2$, the space $\mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ is not a Banach space (it is not a linear space) but it is a complete metric space for the distance

$$d_{\mathbf{\Omega}^p([0,T],\mathbb{R}^d)}(x,y) = \sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dy^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}.$$

The structure of $\mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ will be better understood in the next lectures, but let us remind that if $1 \leq p < 2$, then $\mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ is the closure of $C^{1-var}([0,T],\mathbb{R}^d)$ inside $C^{p-var}([0,T],\mathbb{R}^d)$ for the variation distance it is therefore what we denoted $C^{0,p-var}([0,T],\mathbb{R}^d)$. As a corollary we deduce

Proposition 2.10. *Let $1 \leq p < 2$. Then $x \in \mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ if and only if*

$$\lim_{\delta \to 0} \sup_{\Pi \in \mathcal{D}[s,t], |\Pi| \leq \delta} \sum_{k=0}^{n-1} \| x(t_{k+1}) - x(t_k) \|^p = 0,$$

where $\mathcal{D}[s,t]$ is the set of subdivisions of $[s,t]$. In particular, for $p < q < 2$,

$$C^{q-var}([0,T],\mathbb{R}^d) \subset \mathbf{\Omega}^p([0,T],\mathbb{R}^d).$$

3. Rough linear differential equations

In this lecture we define solutions of linear differential equations driven by p-rough paths, $p \geq 1$ and present the Lyons' continuity theorem in this setting. Let $x \in \mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ be a p-rough path with truncated signature

$$\sum_{k=0}^{[p]} \int_{\Delta^k[s,t]} dx^{\otimes k},$$

and let $x_n \in C^{1-var}([0,T],\mathbb{R}^d)$ be an approximating sequence such that

$$\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} - \int dx_n^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[0,T]} \to 0.$$

Let us consider matrices $M_1, \cdots, M_d \in \mathbb{R}^{n\times n}$. We have the following theorem:

Theorem 3.1. *Let $y_n : [0,T] \to \mathbb{R}^n$ be the solution of the differential equation*

$$y_n(t) = y(0) + \sum_{i=1}^{d} \int_0^t M_i y_n(s) dx_n^i(s).$$

Then, when $n \to \infty$, y_n converges in the p-variation distance to some $y \in C^{p-var}([0,T],\mathbb{R}^n)$. y is called the solution of the rough differential equation

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t M_i y(s) dx^i(s).$$

Proof. It is a classical result that the solution of the equation

$$y_n(t) = y(0) + \sum_{i=1}^{d} \int_0^t M_i y_n(s) dx_n^i(s),$$

can be expanded as the convergent Volterra series:

$$y_n(t) = y_n(s) + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} M_{i_1}\cdots M_{i_k} \left(\int_{\Delta^k[s,t]} dx_n^I \right) y_n(s).$$

Therefore, in particular, for $n, m \geq 0$,

$$y_n(t) - y_p(t) = \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} M_{i_1}\cdots M_{i_k} \left(\int_{\Delta^k[0,t]} dx_n^I - \int_{\Delta^k[0,t]} dx_p^I \right) y(0),$$

which implies that

$$\|y_n(t) - y_m(t)\| \leq \sum_{k=1}^{+\infty} M^k \left\| \int_{\Delta^k[0,t]} dx_n^{\otimes k} - \int_{\Delta^k[0,t]} dx_m^{\otimes k} \right\| \|y(0)\|$$

with $M = \max\{\|M_1\|, \cdots, \|M_d\|\}$. From the theorem of the previous lecture, there exists a constant $C \geq 0$ depending only on p and

$$\sup_n \sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}$$

such that for $k \geq 1$ and n, m big enough:

$$\left\| \int_{\Delta^k[0,\cdot]} dx_n^{\otimes k} - \int_{\Delta^k[0,\cdot]} dx_m^{\otimes k} \right\|_{p-var,[0,T]} \leq \left(\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right) \frac{C^k}{\left(\frac{k}{p}\right)!}.$$

As a consequence, there exists a constant $\tilde{C}$ such that for n, m big enough:

$$\|y_n(t) - y_m(t)\| \leq \tilde{C} \sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}.$$

This already proves that y_n converges in the supremum topology to some y. We now have

$$(y_n(t)-y_n(s))-(y_m(t)-y_m(s)) = \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} M_{i_1} \cdots M_{i_k} \left(\int_{\Delta^k[s,t]} dx_n^I y_n(s) - \int_{\Delta^k[s,t]} dx_m^I y_m(s) \right),$$

and we can bound

$$\begin{aligned}
&\left\| \int_{\Delta^k[s,t]} dx_n^I y_n(s) - \int_{\Delta^k[s,t]} dx_m^I y_m(s) \right\| \\
\leq &\left\| \int_{\Delta^k[s,t]} dx_n^I \right\| \|y_n(s) - y_m(s)\| + \|y_m(s)\| \left\| \int_{\Delta^k[s,t]} dx_n^I - \int_{\Delta^k[s,t]} dx_m^I \right\| \\
\leq &\left\| \int_{\Delta^k[s,t]} dx_n^I \right\| \|y_n - y_m\|_{\infty,[0,T]} + \|y_m\|_{\infty,[0,T]} \left\| \int_{\Delta^k[s,t]} dx_n^I - \int_{\Delta^k[s,t]} dx_m^I \right\|
\end{aligned}$$

From the theorems of the previous lectures, there exists a constant $C \geq 0$, depending only on p and

$$\sup_n \sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}$$

such that for $k \geq 1$ and n, m big enough

$$\left\| \int_{\Delta^k[s,t]} dx_n^{\otimes k} \right\| \leq \frac{C^k}{\left(\frac{k}{p}\right)!} \omega(s,t)^{k/p}, \quad 0 \leq s \leq t \leq T.$$

$$\left\| \int_{\Delta^k[s,t]} dx_n^{\otimes k} - \int_{\Delta^k[s,t]} dx_m^{\otimes k} \right\| \leq \left(\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes k} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right) \frac{C^k}{\left(\frac{k}{p}\right)!} \omega(s,t)^{k/p},$$

where ω is a control such that $\omega(0,T)=1$. Consequently, there is a constant $\tilde{C}$, such that

$$\begin{aligned}
&\|(y_n(t)-y_n(s))-(y_m(t)-y_m(s))\| \\
\leq &\tilde{C}\left(\|y_n-y_m\|_{\infty,[0,T]}+\sum_{j=1}^{[p]}\left\|\int dx_n^{\otimes j}-\int dx_m^{\otimes k}\right\|_{\frac{p}{j}-var,[0,T]}^{1/j}\right)\omega(s,t)^{1/p}
\end{aligned}$$

This implies the estimate

$$\|y_n-y_m\|_{p-var,[0,T]}\leq \tilde{C}\left(\|y_n-y_m\|_{\infty,[0,T]}+\sum_{j=1}^{[p]}\left\|\int dx_n^{\otimes j}-\int dx_m^{\otimes k}\right\|_{\frac{p}{j}-var,[0,T]}^{1/j}\right)$$

and thus gives the conclusion. $\square$

With just a little more work, it is possible to prove the following stronger result whose proof is let to the reader.

Theorem 3.2. *Let $y_n:[0,T]\to\mathbb{R}^n$ be the solution of the differential equation*

$$y_n(t)=y(0)+\sum_{i=1}^{d}\int_0^t M_i y_n(s)dx_n^i(s).$$

and y be the solution of the rough differential equation:

$$y(t)=y(0)+\sum_{i=1}^{d}\int_0^t M_i y(s)dx^i(s).$$

Then, $y\in\mathbf{\Omega}^p([0,T],\mathbb{R}^d)$ and when $n\to\infty$,

$$\sum_{j=1}^{[p]}\left\|\int dy^{\otimes j}-\int dy_n^{\otimes j}\right\|_{\frac{p}{j}-var,[0,T]}^{1/j}\to 0.$$

We can get useful estimates for solutions of rough differential equations. For that, we need the following analysis lemma:

Proposition 3.3. *For $x\geq 0$ and $p\geq 1$,*

$$\sum_{k=0}^{+\infty}\frac{x^k}{\left(\frac{k}{p}\right)!}\leq pe^{x^p}.$$

Proof. For $\alpha\geq 0$, we denote

$$E_\alpha(x)=\sum_{k=0}^{+\infty}\frac{x^k}{(k\alpha)!}.$$

This is a special function called the Mittag-Leffler function. From the binomial inequality

$$\begin{aligned} E_\alpha(x)^2 &= \sum_{k=0}^{+\infty} \left(\sum_{j=0}^{k} \frac{1}{(j\alpha)!\,((k-j)\alpha)!} \right) x^k \\ &\le \frac{1}{\alpha} \sum_{k=0}^{+\infty} 2^{\alpha k} \frac{x^k}{(k\alpha)!} = \frac{1}{\alpha} E_\alpha(2^\alpha x). \end{aligned}$$

Thus we proved

$$E_\alpha(x) \le \frac{1}{\alpha^{1/2}} E_\alpha(2^\alpha x)^{1/2}.$$

Iterating this inequality, k times we obtain

$$E_\alpha(x) \le \frac{1}{\alpha^{\sum_{j=1}^{k} \frac{1}{2^j}}} E_\alpha(2^{\alpha k} x)^{1/(2k)}.$$

It is known (and not difficult to prove) that

$$E_\alpha(x) \sim_{x\to\infty} \frac{1}{\alpha} e^{x^{1/\alpha}}.$$

By letting $k \to \infty$ we conclude

$$E_\alpha(x) \le \frac{1}{\alpha} e^{x^{1/\alpha}}.$$

□

This estimate provides the following result:

Proposition 3.4. *Let y be the solution of the rough differential equation:*

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t M_i y(s) dx^i(s).$$

Then, there exists a constant C depending only on p such that for $0 \le t \le T$,

$$\|y(t)\| \le p\|y(0)\| e^{CM\left(\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[0,t]}\right)^p},$$

where $M = \max\{\|M_1\|, \cdots, \|M_d\|\}$.

Proof. We have

$$y(t) = y(0) + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} M_{i_1} \cdots M_{i_k} \left(\int_{\Delta^k[0,t]} dx_n^I \right) y(0).$$

Thus we obtain

$$y(t) \le \left(1 + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\cdots,i_k)} M^k \left\| \int_{\Delta^k[0,t]} dx_n^I \right\| \right) \|y(0)\|,$$

and we conclude by using estimates on iterated integrals of rough paths together with the previous lemma. □

4. The Chen-Strichartz expansion formula

The next few lectures will be devoted to the construction of the so-called geometric rough paths. These paths are the lifts of the p-rough paths in the free nilpotent Lie group of order p. The construction which is of algebraic and geometric nature will give a clear understanding and description of the space of rough paths. The starting point of the geometric rough path construction is the algebraic study of the signature. We present first the results for continuous paths with bounded variation because the extension to p-rough paths is more or less trivial.

Let us first remind that if $x \in C^{1-var}([0,T],\mathbb{R}^d)$, then the signature of x is defined as the formal series

$$\begin{aligned}\mathfrak{S}(x)_{s,t} &= 1 + \sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \left(\int_{s\le t_1 \le \dots \le t_k \le t} dx_{t_1}^{i_1} \cdots dx_{t_k}^{i_k} \right) X_{i_1} \cdots X_{i_k} \\ &= 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[0,T]} dx^{\otimes k}.\end{aligned}$$

If the indeterminates $X_1, \cdots, X_d$ commute (that is if we work in the commutative algebra of formal series), then the signature of a path admits a very nice representation.

Indeed, let us denote by $\mathcal{S}_k$ the group of the permutations of the index set $\{1,...,k\}$ and if $\sigma \in \mathcal{S}_k$, we denote for a word $I = (i_1,...,i_k)$, $\sigma \cdot I$ the word $(i_{\sigma(1)},...,i_{\sigma(k)})$. By commuting $X_1, \cdots, X_d$ we get

$$\mathfrak{S}(x)_{s,t} = 1 + \sum_{k=1}^{+\infty} \sum_{I=(i_1,\dots,i_k)} X_{i_1}...X_{i_k} \left(\frac{1}{k!} \sum_{\sigma\in\mathcal{S}_k} \int_{\Delta^k[s,t]} dx^{\sigma\cdot I} \right).$$

Since

$$\sum_{\sigma\in\mathcal{S}_k} \int_{\Delta^k[s,t]} dx^{\sigma\cdot I} = (x^{i_1}(t) - x^{i_1}(s)) \cdots (x^{i_k}(t) - x^{i_k}(s)),$$

we deduce,

$$\begin{aligned}\mathfrak{S}(x)_t &= 1 + \sum_{k=1}^{+\infty} \frac{1}{k!} \sum_{I=(i_1,\dots,i_k)} X_{i_1} \cdots X_{i_k} (x^{i_1}(t) - x^{i_1}(s)) \cdots (x^{i_k}(t) - x^{i_k}(s)) \\ &= \exp\left(\sum_{i=0}^{d} (x^i(t) - x^i(s)) X_i \right),\end{aligned}$$

where the exponential of a formal series Y is, of course, defined as

$$\exp(Y) = \sum_{k=0}^{+\infty} \frac{Y^k}{k!}.$$

As a consequence, the *commutative* signature of a path is simply the exponential of the increments of the path. Of course, the formula is only true in the commutative case. In the general and non-commuting case, it is remarkable that there exists a nice formula that expresses the signature as the exponential of a quite explicit series which turns out to be a Lie series (a notion defined below). We need to introduce first a few notations.

We define the Lie bracket between two elements U and V of $\mathbb{R}[[X_1, \cdots, X_d]]$ by

$$[U, V] = UV - VU.$$

Moreover, if $I = (i_1, ..., i_k) \in \{1, \cdots, d\}^k$ is a word, we denote by X_I the iterated Lie bracket which is defined by

$$X_I = [X_{i_1}, [X_{i_2}, ..., [X_{i_{k-1}}, X_{i_k}]...].$$

Theorem 4.1. *[Chen-Strichartz expansion theorem] If $x \in C^{1-var}([0, T], \mathbb{R}^d)$, then*

$$\mathfrak{S}(x)_{s,t} = \exp\left(\sum_{k\geq 1}\sum_{I\in\{1,\cdots,d\}^k} \Lambda_I(x)_{s,t} X_I\right), \ 0 \leq s \leq t \leq T,$$

where for $k \geq 1$, $I \in \{1, \cdots, d\}^k$:

- *$\mathcal{S}_k$ is the set of the permutations of $\{1, \cdots, k\}$;*
- *If $\sigma \in \mathcal{S}_k$, $e(\sigma)$ is the cardinality of the set*

$$\{j \in \{1, \cdots, k-1\}, \sigma(j) > \sigma(j+1)\},$$

- $$\Lambda_I(x)_{s,t} = \sum_{\sigma\in\mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{\Delta^k[s,t]} dx^{\sigma^{-1}\cdot I}.$$

Remark 4.2. *The first terms in the Chen-Strichartz formula are:*

(1)

$$\sum_{I=(i_1)} \Lambda_I(x)_{s,t} X_I = \sum_{k=1}^{d} (x^i(t) - x^i(s)) X_i;$$

(2)

$$\sum_{I=(i_1,i_2)} \Lambda_I(x)_{s,t} X_I = \frac{1}{2} \sum_{1\leq i<j\leq d} [X_i, X_j] \int_s^t x^i(u) dx^j(u) - x^j(u) dx^i(u).$$

The proof proceeds in several steps. To simplify a little the notations we will assume $s = 0$, $t = T$ and $x(0) = 0$. The idea is to prove first the result when the path x is piecewise linear that is

$$x(t) = x(t_i) + a_i(t - t_i)$$

on the interval $[t_i, t_{i+1})$ where $0 = t_0 \leq t_1 \leq \cdots \leq t_N = T$. And, then, we will use a limiting argument.

The key point here is the multiplicativity property for the signature that already was pointed out in a previous lecture: For $0 \leq s \leq t \leq u \leq T$,

$$\mathfrak{S}(x)_{s,u} = \mathfrak{S}(x)_{s,t}\mathfrak{S}(x)_{t,u}.$$

By using inductively the multiplicative property, we obtain

$$\mathfrak{S}(x)_{0,T} = \prod_{n=0}^{N-1} \left(\mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...i_k)} X_{i_1}...X_{i_k} \int_{\Delta^k[t_n,t_{n+1}]} dx^I \right)$$

Since, on $[t_n, t_{n+1})$,

$$dx(t) = a_n dt,$$

we have

$$\int_{\Delta^k[t_n,t_{n+1}]} dx^I = a_n^{i_1} \cdots a_n^{i_k} \int_{\Delta^k[t_n,t_{n+1}]} dt_{i_1} \cdots dt_{i_k} = a_n^{i_1} \cdots a_n^{i_k} \frac{(t_{n+1}-t_n)^k}{k!}.$$

Therefore

$$\begin{aligned}\mathfrak{S}(x)_{0,T} &= \prod_{n=0}^{N-1} \left(\mathbf{1} + \sum_{k=1}^{+\infty} \sum_{I=(i_1,...i_k)} X_{i_1}...X_{i_k} a_n^{i_1} \cdots a_n^{i_k} \frac{(t_{n+1}-t_n)^k}{k!} \right) \\ &= \prod_{n=0}^{N-1} \exp\left((t_{n+1}-t_n) \sum_{i=0}^{d} a_n^i X_i \right).\end{aligned}$$

We now use the Baker-Campbell-Hausdorff-Dynkin formula that gives a quite explicit formula for the product of exponentials of non commuting variables:

Proposition 4.3 (Baker-Campbell-Hausdorff-Dynkin formula)**.** *If $y_1, \cdots, y_N \in \mathbb{R}^d$ then,*

$$\prod_{n=1}^{N} \exp\left(\sum_{i=1}^{d} y_n^i X_i \right) = \exp\left(\sum_{k \geq 1} \sum_{I \in \{1,...,d\}^k} \beta_I(y_1, \cdots, y_N) X_I \right),$$

where for $k \geq 1$, $I \in \{1,...,d\}^k$:

$$\beta_I(y_1, \cdots, y_N) = \sum_{\sigma \in \mathcal{S}_k} \sum_{1=j_0 \leq j_1 \leq \cdots \leq j_{N-1} \leq k} \frac{(-1)^{e(\sigma)}}{j_1! \cdots j_{N-1}! k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \prod_{\nu=1}^{N} y_\nu^{\sigma^{-1}(i_{j_{\nu-1}+1})} \cdots y_\nu^{\sigma^{-1}(i_{j_\nu})}.$$

We get therefore:

$$\mathfrak{S}(x)_{0,T} = \exp\left(\sum_{k \geq 1} \sum_{I \in \{1,...,d\}^k} \beta_I(t_1 a_0, \cdots, (t_N - t_{N-1}) a_{N-1}) X_I \right).$$

It is finally an exercise to check, by using the Chen relations, that:

$$\beta_I(t_1 a_0, \cdots, (t_N - t_{N-1}) a_{N-1}) = \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{\Delta^k[0,T]} dx^{\sigma^{-1} \cdot I}.$$

We conclude that if x is piecewise linear then the formula

$$\mathfrak{S}(x)_{s,t} = \exp \left(\sum_{k \geq 1} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_{s,t} X_I \right), \ 0 \leq s \leq t \leq T$$

holds. Finally, if $x \in C^{1-var}([0,T], \mathbb{R}^d)$, then we can consider the sequence x_n of linear interpolations along a subdivision of $[0,T]$ whose mesh goes to 0. For this sequence, all the iterated integrals $\int_{\Delta^k[0,T]} dx_n^I$ will converge to $\int_{\Delta^k[0,T]} dx^I$ (see for instance the proposition 2.7 in the book by Friz-Victoir) and the result follows.

5. Magnus expansion

In the previous lecture, we proved the Chen's expansion formula which establishes the fact that the signature of a path is the exponential of a Lie series. This expansion is of course formal but analytically makes sense in a number of situations that we now describe. The first case of study are linear equations.

Let us consider matrices $M_1, \cdots, M_d \in \mathbb{R}^{n \times n}$ and let $y_n : [0,T] \to \mathbb{R}^n$ be the solution of the differential equation

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t M_i y(s) dx^i(s),$$

where $x \in C^{1-var}([0,T], \mathbb{R}^d)$. The solution y admits a representation as an absolutely convergent Volterra series

$$y(t) = y(0) + \sum_{k=1}^{+\infty} \sum_{I=(i_1, \cdots, i_k)} M_{i_1} \cdots M_{i_k} \left(\int_{\Delta^k[0,t]} dx^I \right) y(0).$$

The formal analogy between this expansion and the signature leads to the following result:

Proposition 5.1. *There exists $\tau > 0$ such that for $0 \leq t \leq \tau$,*

$$y(t) = \exp \left(\sum_{k \geq 1} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t M_I \right) y(0),$$

where

$$M_I = [M_{i_1}, [M_{i_2}, ..., [M_{i_{k-1}}, M_{i_k}]...],$$

is the iterated Lie bracket and

$$\Lambda_I(x)_t = \sum_{\sigma \in \mathcal{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{\Delta^k[0,t]} dx^{\sigma^{-1} \cdot I}.$$

PROOF. We only give the sketch of the proof. Details can be found in this paper by Strichartz. First, we observe that a combinatorial argument shows that

$$\sum_{\sigma \in \mathcal{S}_k} \frac{1}{\begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \le \frac{C}{2^k} k! \sqrt{k}.$$

On the other hand, we have the estimate

$$\left| \int_{\Delta^k[0,t]} dx^{\sigma^{-1} \cdot I} \right| \le \int_{\Delta^k[0,t]} \| dx^{\sigma^{-1} \cdot I} \| \le \frac{1}{k!} \|x\|^k_{1-var,[0,t]}.$$

As a consequence, we obtain

$$|\Lambda_I(x)_t| \le \frac{C}{2^k k^{3/2}} \|x\|^k_{1-var,[0,t]}.$$

For the matrix norm we have the estimate

$$\|M_I\| \le C^k,$$

so we conclude that for some constant $\tilde{C}$,

$$\left\| \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t M_I \right\| \le \frac{\tilde{C}^k}{k^{3/2}} \|x\|^k_{1-var,[0,t]}.$$

We deduce that if τ is such that $\|x\|_{1-var,[0,\tau]} < \frac{1}{\tilde{C}}$, then the series

$$\sum_{k \ge 1} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t M_I$$

is absolutely convergent on the interval $[0, \tau]$. At this point, we can observe that the Chen's expansion formula is a purely algebraic statement, thus expanding the exponential

$$\exp \left(\sum_{k \ge 1} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t M_I \right) y(0)$$

and rearranging the terms leads to

$$y(0) + \sum_{k=1}^{+\infty} \sum_{I=(i_1, \cdots, i_k)} M_{i_1} \cdots M_{i_k} \left(\int_{\Delta^k[0,t]} dx^I \right) y(0)$$

which is equal to $y(t)$. □

Another framework, close to this linear case, in which the Chen's expansion makes sense are Lie groups. Let $\mathbb{G}$ be a Lie group acting on $\mathbb{R}^d$. Let us denote by $\mathfrak{g}$ the Lie algebra of $\mathbb{G}$. Elements of $\mathfrak{g}$ can be seen as vector fields on $\mathbb{R}^d$. Indeed, for $X \in \mathfrak{g}$, we can define

$$X(x) = \lim_{t \to 0} \frac{e^{tX}(x) - x}{t},$$

where e^{tX} is the exponential mapping on the Lie group $\mathbb{G}$. With this identification, it is easily checked that the Lie bracket in the Lie algebra coincides with the Lie bracket of vector fields and that the exponential map e^{tX} in the group corresponds to the flow generated by the vector field X. As above we get then the following result:

Proposition 5.2. *Let $V_1, \cdots, V_d \in \mathfrak{g}$ and $x \in C^{1-var}([0,T], \mathbb{R}^d)$. Let us consider the differential equation*

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s).$$

There exists $\tau > 0$ such that for $0 \le t \le \tau$,

$$y(t) = \exp\left(\sum_{k \ge 1} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t V_I \right) y(0).$$

A special case will be of interest for us: The case where the Lie group $\mathbb{G}$ is nilpotent. Let us recall that a Lie group $\mathbb{G}$ is said to be nilpotent of order N if every bracket of length greater or equal to $N+1$ is 0. In that case, the sum in the exponential is finite and the representation is then of course valid on the whole time interval $[0,T]$.

6. Free Carnot groups

We introduce here the notion of Carnot group, which is the correct structure to understand the algebra of the iterated integrals of a path up to a given order. It is worth mentioning that these groups play a fundamental role in sub-Riemannian geometry as they appear as the tangent cones to sub-Riemannian manifolds.

Definition 6.1. *A Carnot group of step (or depth) N is a simply connected Lie group $\mathbb{G}$ whose Lie algebra can be written*

$$\mathcal{V}_1 \oplus ... \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \text{ for } s > N.$$

We have some basic examples of Carnot groups.

Example 6.2. *The group $\left(\mathbb{R}^d, +\right)$ is the only commutative Carnot group.*

Example 6.3. *Consider the set $\mathbb{H}_n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law*

$$(x, \alpha) \star (y, \beta) = \left(x + y, \alpha + \beta + \frac{1}{2}\omega(x,y) \right),$$

where ω is the standard symplectic form on $\mathbb{R}^{2n}$, that is

$$\omega(x,y) = x^t \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix} y.$$

On $\mathfrak{h}_n$ the Lie bracket is given by

$$[(x,\alpha),(y,\beta)] = (0, \omega(x,y)),$$

and it is easily seen that

$$\mathfrak{h}_n = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_1 = \mathbb{R}^{2n} \times \{0\}$ *and* $\mathcal{V}_2 = \{0\} \times \mathbb{R}$*. Therefore* $\mathbb{H}_n$ *is a Carnot group of depth 2.*

The Carnot group $\mathbb{G}$ is said to be free if $\mathfrak{g}$ is isomorphic to the nilpotent free Lie algebra with d generators. In that case, $\dim \mathcal{V}_j$ is the number of Hall words of length j in the free algebra with d generators. A combinatorial argument shows then that:

$$\dim \mathcal{V}_j = \frac{1}{j} \sum_{i|j} \mu(i) d^{\frac{j}{i}}, \ j \leq N,$$

where μ is the Möbius function. A consequence from this is that when $N \to +\infty$,

$$\dim \mathfrak{g} \sim \frac{d^N}{N}.$$

The free Carnot groups are the ones that will be the most relevant for us, so from now on, we will restrict our attention to them.

Let $\mathbb{G}$ be a free Carnot group of step N. Notice that the vector space $\mathcal{V}_1$, which is called the basis of $\mathbb{G}$, Lie generates $\mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}$. Since $\mathbb{G}$ is step N nilpotent and simply connected, the exponential map is a diffeomorphism and the Baker-Campbell-Hausdorff formula therefore completely characterizes the group law of $\mathbb{G}$ because for $U, V \in \mathfrak{g}$,

$$\exp U \exp V = \exp\left(P(U, V)\right)$$

for some universal Lie polynomial P whose first terms are given by

$$\begin{aligned} P(U,V) &= U + V + \tfrac{1}{2}[U,V] + \tfrac{1}{12}[[U,V],V] - \tfrac{1}{12}[[U,V],U] \\ &\quad - \tfrac{1}{48}[V,[U,[U,V]]] - \tfrac{1}{48}[U,[V,[U,V]]] + \cdots . \end{aligned}$$

(see Appendix B for an explicit formula). On $\mathfrak{g}$ we can consider the family of linear operators $\delta_t : \mathfrak{g} \to \mathfrak{g}$, $t \geq 0$ which act by scalar multiplication t^i on $\mathcal{V}_i$. These operators are Lie algebra automorphisms due to the grading. The maps δ_t induce Lie group automorphisms $\Delta_t : \mathbb{G} \to \mathbb{G}$ which are called the canonical dilations of $\mathbb{G}$

It is an interesting fact that every free Carnot group of step N is isomorphic to some $\mathbb{R}^m$ endowed with a polynomial group law. Indeed, let $X_1, \cdots, X_d$ be a basis of $\mathcal{V}_1$. From the Hall-Witt theorem we can construct a basis of $\mathfrak{g}$ which is adapted to the grading

$$\mathfrak{g} = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N,$$

and such that every element of this basis is an iterated bracket of the X_i's. Such basis, which can be made quite explicit, will be referred to as a Hall basis over $X_1, \cdots, X_d$. Let $\mathcal{B}$ be such a basis. For $X \in \mathfrak{g}$, let $[X]_{\mathcal{B}}$ be the coordinate vector of X in the basis $\mathcal{B}$. If we denote by m the dimension of $\mathfrak{g}$, we see that we can define a group law $\star$ on $\mathbb{R}^m$ by the requirement that for $X, Y \in \mathfrak{g}$,

$$[X]_{\mathcal{B}} \star [Y]_{\mathcal{B}} = [P_N(X,Y)]_{\mathcal{B}} = [\ln(e^X e^Y)]_{\mathcal{B}}.$$

It is then clear that $(\mathbb{R}^m, \star)$ is a Carnot group of step N whose Lie bracket is given by:

$$[[X]_{\mathcal{B}}, [Y]_{\mathcal{B}}] = [[X,Y]]_{\mathcal{B}}.$$

Therefore, every free Carnot group of step N such that $\dim \mathcal{V}_1 = d$ is isomorphic to $(\mathbb{R}^m, \star)$. Another representation of the free Carnot group of step N which is particularly adapted to rough paths theory is given in the framework of formal series. As before, let us denote by $\mathbb{R}[[X_1, \cdots, X_d]]$ the set formal series. Let us denote by $\mathbb{R}_N[X_1, \cdots, X_d]$ the set of truncated series at order N, that is $\mathbb{R}[[X_1, \cdots, X_d]]$ quotiented by $X_{i_1} \cdots X_{i_k} = 0$ if $k \geq N+1$. In this context, the free nilpotent Lie algebra of order N can be identified with the Lie algebra generated by $X_1, \cdots, X_d$ inside $\mathbb{R}_N[X_1, \cdots, X_d]$, where the bracket is of course given by the anticommutator. This representation of the free nilpotent Lie algebra of depth N shall be denoted by $\mathfrak{g}_N(\mathbb{R}^d)$ in the sequel of the course. The free nilpotent group of step can then be represented as $\mathbb{G}_N(\mathbb{R}^d) = \exp(\mathfrak{g}_N(\mathbb{R}^d))$ where the exponential map is the usual exponential of formal series.

We are now ready for the definition of the lift of a path in $\mathbb{G}_N(\mathbb{R}^d)$.

Definition 6.4. *Let $x \in C^{1-var}([0,T], \mathbb{R}^d)$. The $\mathbb{G}_N(\mathbb{R}^d)$ valued path*

$$\sum_{k=0}^{N} \int_{\Delta^k[0,t]} dx^{\otimes k}, \quad 0 \leq t \leq T,$$

is called the lift of x in $\mathbb{G}_N(\mathbb{R}^d)$ and will be denoted by $S_N(x)$.

It is worth noticing that $S_N(x)$ is indeed valued in $\mathbb{G}_N(\mathbb{R}^d)$ because from the Chen's expansion formula:

$$S_N(x)(t) = \exp\left(\sum_{k=1}^{N} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_t X_I \right),$$

where the notations have been introduced before. The multiplicativity property of the signature also immediately implies that for $s \leq t$,

$$S_N(x)(t) = S_N(x)(s) \exp\left(\sum_{k=1}^{N} \sum_{I \in \{1, \cdots, d\}^k} \Lambda_I(x)_{s,t} X_I \right).$$

7. The Carnot-Carathéodory distance

In this Lecture we introduce a canonical distance on a Carnot group. This distance is naturally associated to the sub-Riemannian structure which is carried out by a Carnot group. It plays a fundamental role in the rough paths topology. Let $\mathbb{G}_N(\mathbb{R}^d)$ be the free Carnot group over $\mathbb{R}^d$. Remember that if $x \in C^{1-var}([0,T], \mathbb{R}^d)$, then we denote by $S_N(x)$ the lift of x in $\mathbb{G}_N(\mathbb{R}^d)$. The first important concept is the notion of horizontal curve.

Definition 7.1. *A curve $y : [0,1] \to \mathbb{G}_N(\mathbb{R}^d)$ is said to be horizontal if there exists $x \in C^{1-var}([0,T], \mathbb{R}^d)$ such that $y = S_N(x)$.*

It is remarkable that any two points of $\mathbb{G}_N(\mathbb{R}^d)$ can be connected by a horizontal curve.

Theorem 7.2. *Given two points g_1 and $g_2 \in \mathbb{G}_N(\mathbb{R}^d)$, there is at least one $x \in C^{1-var}([0,T], \mathbb{R}^d)$ such that $g_1 S_N(x)(1) = g_2$.*

PROOF. Let us denote G the subgroup of diffeomorphisms $\mathbb{G}_N(\mathbb{R}^d) \to \mathbb{G}_N(\mathbb{R}^d)$ generated by the one-parameter subgroups corresponding to $\mathbf{e}_1, \cdots, \mathbf{e}_d$. The Lie algebra of G can be identified with the Lie algebra generated by $X_1, \cdots, X_d$, i.e. $\mathfrak{g}_N(\mathbb{R}^d)$. We deduce that G can be identified with $\mathbb{G}_N(\mathbb{R}^d)$ itself, so that it acts transitively on $\mathbb{G}$. It means that for every $x \in \mathbb{G}_N(\mathbb{R}^d)$, the map $G \to \mathbb{G}_N(\mathbb{R}^d)$, $g \to g(x)$ is surjective. Thus, every two points in $\mathbb{G}$ can be joined by a piecewise smooth horizontal curve where each piece is a segment of an integral curve of one of the vector fields X_i. $\square$

Remark 7.3. *In the above proof, the horizontal curve constructed to join the two points is not smooth. Nevertheless, it can be shown that it is always possible to connect two points with a smooth horizontal curve.*

This theorem is a actually a very special case of the so-called Chow-Rashevski theorem which is one of the cornerstones of sub-Riemannian geometry. We now are ready for the definition of the Carnot-Carathéodory distance.

Definition 7.4. *For $g_1, g_2 \in \mathbb{G}_N(\mathbb{R}^d)$, we define*

$$d(g_1, g_2) = \inf_{\mathcal{S}(g_1,g_2)} \|x\|_{1-var,[0,1]},$$

where

$$\mathcal{S}(g_1, g_2) = \{x \in C^{1-var}([0,1], \mathbb{R}^d), g_1 S_N(x)(1) = g_2\}.$$

$d(g_1, g_2)$ is called the Carnot-Carathéodory distance between g_1 and g_2.

The first thing to prove is that d is indeed a distance.

Lemma 7.5. *The Carnot-Carathéodory distance is indeed a distance.*

PROOF. The symmetry and the triangle inequality are easy to check and we let the reader find the arguments. The last thing to prove is that $d(g_1, g_2) = 0$ implies $g_1 = g_2$. From the definition of d it clear that $d_R \leq d$ where d_R is the Riemmanian measure on $\mathbb{G}_N(\mathbb{R}^d)$. It follows that $d(g_1, g_2) = 0$ implies $g_1 = g_2$. $\square$

We then observe the following properties of d:

Proposition 7.6.

- *For $g_1, g_2 \in \mathbb{G}_N(\mathbb{R}^d)$,*

$$d(g_1, g_2) = d(g_2, g_1) = d(0, g_1^{-1} g_2).$$

- *Let $(\Delta_t)_{t \geq 0}$ be the one parameter family of dilations on $\mathbb{G}_N(\mathbb{R}^d)$. For $g_1, g_2 \in \mathbb{G}_N(\mathbb{R}^d)$, and $t \geq 0$,*

$$d(\Delta_t g_1, \Delta_t g_2) = t d(g_1, g_2).$$

PROOF. The first part of the proposition stems from the fact that for every $x \in C^{1-var}([0,T], \mathbb{R}^d)$, $S_N(x)^{-1} = S_N(-x)$, so that $g_1 S_N(x)(1) = g_2$ is equivalent to $g_2 S_N(-x)(1) = g_1$ which also equivalent to $S_N(x)(1) = g_1^{-1} g_2$. For the second part, we observe that for $t \geq 0$, $\Delta_t S_N(x) = S_N(tx)$. $\square$

The Carnot-Carathéodory distance is pretty difficult to explicitly compute in general. It is often much more convenient to estimate using a so-called homogeneous norm.

Definition 7.7. *A homogeneous norm on* $\mathbb{G}_N(\mathbb{R}^d)$ *is a continuous function* $\| \cdot \| \colon \mathbb{G}_N(\mathbb{R}^d) \to [0, +\infty)$, *such that:*

(1) $\| \Delta_t x \| = t \| x \|$, $t \geq 0$, $x \in \mathbb{G}_N(\mathbb{R}^d)$;
(2) $\| x^{-1} \| = \| x \|$, $x \in \mathbb{G}_N(\mathbb{R}^d)$;
(3) $\| x \| = 0$ *if and only if* $x = 0$.

It turns out that the Carnot-Carathéodory distance is equivalent to any homogeneous norm in the following sense:

Theorem 7.8. *Let* $\| \cdot \|$ *be a homogeneous norm on* $\mathbb{G}_N(\mathbb{R}^d)$. *There exist two positive constants* C_1 *and* C_2 *such that for every* $x, y \in \mathbb{G}_N(\mathbb{R}^d)$,

$$A \|x^{-1} y\| \leq d(x,y) \leq B \|x^{-1} y\|.$$

Proof. By using the left invariance of d, it is of course enough to prove that for every $x \in \mathbb{G}_N(\mathbb{R}^d)$,

$$A \|x\| \leq d(0,x) \leq B \|x\|.$$

We first prove that the function $x \to d(0,x)$ is bounded on compact sets (of the Riemannian topology of the Lie group $\mathbb{G}_N(\mathbb{R}^d)$). As we have seen before, every $x \in \mathbb{G}_N(\mathbb{R}^d)$ can be written as a product:

$$x = \prod_{i=1}^{N} e^{t_i X_{k_i}}.$$

From the very definition of the distance, we have then

$$d(0,x) \leq d\left(0, \prod_{i=1}^{N} e^{t_i X_{k_i}}\right) \leq \sum_{i=1}^{N} |t_i|.$$

It is not difficult to see that $\sum_{i=1}^{N} |t_i|$ can uniformly be bounded on compact sets, therefore $d(0,x)$ is bounded on compact sets. Consider now the compact set

$$\mathbf{K} = \{x \in \mathbb{G}_N(\mathbb{R}^d), \|x\| = 1\}.$$

Since $d(0,x)$ is bounded on K, we deduce that there exists a constant B such that for every $x \in \mathbf{K}$,

$$d(0,x) \leq B.$$

Since $d_R \leq d$, where d_R is the Riemannian distance, we deduce that there exists a constant A such that for every $x \in \mathbf{K}$,

$$d(0,x) \geq A.$$

Now, for every $x \in \mathbb{G}_N(\mathbb{R}^d)$, $x \neq 0$, we deduce that

$$A \leq d\left(0, \Delta_{1/\|x\|} x\right) \leq B.$$

This yields the expected result. □

Let us now finally give an example of a homogeneous norm which is particularly adapted to rough paths theory. Write the stratification of $\mathfrak{g}_N(\mathbb{R}^d)$ as:

$$\mathfrak{g}_N(\mathbb{R}^d) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N$$

and denote by π_i the projection onto $\mathcal{V}_i$. Let us denote by $\|\cdot\|$ the norm on $\mathfrak{g}_N(\mathbb{R}^d)$ that comes from the norm on formal series. Then, it is easily checked that

$$\rho(g) = \sum_{i=1}^{N} \|\pi_i(g)\|^{1/i}$$

is an homogeneous norm on $\mathbb{G}_N(\mathbb{R}^d)$. This homogeneous norm is particulary adapted to the study of paths because if $x \in C^{1-var}([0,T],\mathbb{R}^d)$, then one has:

$$\rho\left((S_N(x)(s))^{-1}S_N(x)(t)\right) = \sum_{k=1}^{N} \left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\|^{1/k}$$

We finally quote the following result, not difficult to prove which is often referred to as the ball-box estimate.

Proposition 7.9. *There exists a constant C such that for every $x, y \in \mathbb{G}_N(\mathbb{R}^d)$,*

$$d(x,y) \le C \max\{\|x-y\|, \|x-y\|^{1/N} \max\{1, d(0,x)^{1-1/N}\}\}.$$

and

$$\|x-y\| \le C \max\{d(x,y)\max\{1, d(0,x)^{N-1}\}, d(x,y)^N\}.$$

In particular, for every compact set $K \subset \mathbb{G}_N(\mathbb{R}^d)$, there is a constant C_K such that for every $x, y \in K$,

$$\frac{1}{C_K}\|x-y\| \le d(x,y) \le C_K \|x-y\|^{1/N}.$$

Proof. See the book by Friz-Victoir, page 152. □

8. Geometric rough paths

Definition 8.1. *A continuous path $x : [s,t] \to \mathbb{G}_N(\mathbb{R}^d)$ is said to have a bounded variation on $[s,t]$, if the 1-variation of x on $[s,t]$, which is defined as*

$$\|x\|_{1-var;[s,t]} := \sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} d(x(t_{k+1}), x(t_k)),$$

is finite, where d is the Carnot-Carathéodory distance on $\mathbb{G}_N(\mathbb{R}^d)$. The space of continuous bounded variation paths $x : [s,t] \to \mathbb{R}^d$, will be denoted by $C^{1-var}([s,t], \mathbb{G}_N(\mathbb{R}^d))$.

The 1-variation distance between $x, y \in C^{1-var}([s,t], \mathbb{G}_N(\mathbb{R}^d))$ is then defined as

$$d_{1-var;[s,t]} = \sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} d(x(t_k)^{-1}x(t_{k+1}), y(t_k)^{-1}y(t_{k+1})).$$

As for the linear case the following proposition is easy to prove:

Proposition 8.2. *Let $x \in C^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d))$. The function $(s,t) \to \|x\|_{1-var,[s,t]}$ is additive, i.e for $0 \le s \le t \le u \le T$,*

$$\|x\|_{1-var,[s,t]} + \|x\|_{1-var,[t,u]} = \|x\|_{1-var,[s,u]},$$

and controls x in the sense that for $0 \le s \le t \le T$,

$$d(x(s), x(t)) \le \|x\|_{1-var,[s,t]}.$$

The function $s \to \|x\|_{1-var,[0,s]}$ is moreover continuous and non decreasing.

We will denote $C_0^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d)$ the space of continuous bounded variation paths that start at 0. It turns out that $C_0^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d))$ is always isometric to $C_0^{1-var}([0,T], \mathbb{R}^d)$.

Theorem 8.3. *For every, $x \in C_0^{1-var}([0,T], \mathbb{R}^d)$, we have*

$$\|S_N(x)\|_{1-var;[0,T]} = \|x\|_{1-var;[0,T]}.$$

Moreover, for every $y \in C_0^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d)$, there exists one and only one $x \in C_0^{1-var}([0,T], \mathbb{R}^d)$ such that

$$y = S_N(x).$$

Proof. Let $x \in C_0^{1-var}([0,T], \mathbb{R}^d)$. From the very definition of the Carnot-Carathéodory distance, for $0 \le s \le t \le T$, we have

$$d(S_N(x)(s), S_N(x)(t)) \ge \|x\|_{1-var,[s,t]}.$$

As a consequence we obtain,

$$\|S_N(x)\|_{1-var;[0,T]} \ge \|x\|_{1-var;[0,T]}.$$

On the other hand, $S_N(x)$ is the solution of the differential equation

$$S_N(x)(t) = \sum_{i=1}^{d} \int_0^t X_i(S_N(x)(s))dx^i(s), \quad 0 \le t \le T.$$

This implies,

$$d\left(S_N(x)(s), S_N(x)(t)\right) \le \int_s^t \|dx(u)\| = \|x\|_{1-var,[s,t]}.$$

Finally, let $y \in C_0^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d)$. Let x be the projection of y onto $\mathbb{R}^d$. From the theorem of equivalence of homogeneous norms, we deduce that x has a bounded variation in $\mathbb{R}^d$. We claim that $y = S_N(x)$. Consider the path $z = yS_N(x)^{-1}$. This is a bounded variation path whose projection on $\mathbb{R}^d$ is 0. We want to prove that it implies that $z = 0$. Denote by z_2 the projection of z onto $\mathbb{G}_2(\mathbb{R}^d)$. Again from the equivalence of homogeneous norms, we see that z_2 has a bounded variation in $\mathbb{G}_2(\mathbb{R}^d)$. Since the projection of z_2 on $\mathbb{R}^d$ is 0, we deduce that z_2 is in the center of $\mathbb{G}_2(\mathbb{R}^d)$, which implies that $z_2(s)^{-1}z_2(t) = z_2(t) - z_2(s)$. From the equivalence of homogeneous norms, we have then

$$d(z_2(s), z_2(t)) \simeq \|z_2(t) - z_2(s)\|^{1/2}.$$

Since z_2 has a bounded variation in $\mathbb{G}_2(\mathbb{R}^d)$, it has thus a 1/2-variation for the Euclidean norm. This implies $z_2 = 0$. Using the same argument inductively shows that for $n \le N$, the projection of z onto $\mathbb{G}_n(\mathbb{R}^d)$ will be 0. We conclude $z = 0$. □

As a conclusion, bounded variation paths in Carnot groups are the lifts of the bounded variation paths in $\mathbb{R}^d$. As we will see, the situation is very different for paths with bounded p-variation when $p \ge 2$.

Definition 8.4. *Let $p \geq 1$. A continuous path $x : [s,t] \to \mathbb{G}_N(\mathbb{R}^d)$ is said to have a bounded p-variation on $[s,t]$, if the p-variation of x on $[s,t]$, which is defined as*

$$\|x\|_{p-var;[s,t]} := \left(\sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} d(x(t_{k+1}), x(t_k))^p \right)^{1/p},$$

is finite. The space of continuous paths $x : [s,t] \to \mathbb{R}^d$ with a p-bounded variation will be denoted by $C^{p-var}([s,t], \mathbb{G}_N(\mathbb{R}^d))$.

The p-variation distance between $x, y \in C^{p-var}([s,t], \mathbb{G}_N(\mathbb{R}^d))$ is then defined as

$$d_{p-var;[s,t]}(x,y) = \left(\sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} d(x(t_k)^{-1}x(t_{k+1}), y(t_k)^{-1}y(t_{k+1}))^p \right)^{1/p}.$$

As for $\mathbb{R}^d$ valued paths, we restrict our attention to $p \geq 1$ because any path with a p-bounded variation, $p < 1$ needs to be constant. We have then the following theorem that extends the previous result. The proof is somehow similar to the previous result, so we let the reader fill the details.

Theorem 8.5. *Let $1 \leq p < 2$. For every $y \in C_0^{p-var}([0,T], \mathbb{G}_N(\mathbb{R}^d))$, there exists one and only one $x \in C_0^{p-var}([0,T], \mathbb{R}^d)$ such that*

$$y = S_N(x).$$

Moreover, we have

$$\|x\|_{p-var;[0,T]} \leq \|S_N(x)\|_{p-var;[0,T]} \leq C\|x\|_{p-var;[0,T]}.$$

For $p \geq 2$, the situation is different as we are going to explain in the next Lectures. This can already be understood by using the estimates that were obtained in a previous Lecture. Indeed, we have the following very important proposition that already shows the connection between p-rough paths and paths with a bounded p-variation in Carnot groups:

Proposition 8.6. *Let $p \geq 1$ and $N \geq [p]$. There exist constants $C_1, C_2 > 0$ such that for every $x \in C_0^{1-var}([0,T], \mathbb{G}_N(\mathbb{R}^d))$,*

$$C_1 \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|_{\frac{p}{j}-var,[s,t]}^{1/j} \right) \leq \|S_N(x)\|_{p-var;[s,t]} \leq C_2 \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|_{\frac{p}{j}-var,[s,t]}^{1/j} \right)$$

Proof. This is a consequence of the theorem about the equivalence of homogeneous norms on Carnot groups. Write the stratification of $\mathfrak{g}_N(\mathbb{R}^d)$ as:

$$\mathfrak{g}_N(\mathbb{R}^d) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_N$$

and denote by π_i the projection onto $\mathcal{V}_i$. Let us denote by $\|\cdot\|$ the norm on $\mathfrak{g}_N(\mathbb{R}^d)$ that comes from the norm on formal series. Then,

$$\rho(g) = \sum_{i=1}^{N} \|\pi_i(g)\|^{1/i}$$

is an homogeneous norm on $\mathbb{G}_N(\mathbb{R}^d)$. Thus, there exist constants $C_1, C_2 > 0$ such that for every $g \in \mathbb{G}_N(\mathbb{R}^d)$,

$$C_1\rho(g) \le d(0,g) \le C_2\rho(g).$$

In particular, we get

$$C_1\rho\left(S_N(x)(s)^{-1}S_N(x)(t)\right) \le d\left(S_N(x)(s), S_N(x)(t)\right) \le C_2\rho\left(S_N(x)(s)^{-1}S_N(x)(t)\right).$$

Let us now observe that

$$\rho\left((S_N(x)(s))^{-1}S_N(x)(t)\right) = \sum_{k=1}^{N} \left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\|^{1/k}$$

and that, from a previous lecture $k \ge [p]$,

$$\left\| \int_{\Delta^k[s,t]} dx^{\otimes k} \right\| \le \frac{C^k}{\left(\frac{k}{p}\right)!} \left(\sum_{j=1}^{[p]} \left\| \int dx^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[s,t]} \right)^k, \quad 0 \le s \le t \le T.$$

The conclusion easily follows. □

In this Lecture, the geometric concepts introduced in the previous lectures are now used to revisit the notion of p-rough path that was introduced before. We will see that using Carnot groups gives a perfect description of the space of p-rough paths through the notion of geometric rough path.

Definition 8.7. *Let $p \ge 1$. An element $x \in C_0^{p-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ is called a geometric p-rough path if there exists a sequence $x_n \in C_0^{1-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ that converges to x in the p-variation distance. The space of geometric p-rough paths will be denoted by $\mathbf{\Omega G}^p([0,T], \mathbb{R}^d)$.*

To have it in mind, we recall the definition of a p-rough path.

Definition 8.8. *Let $p \ge 1$ and $x \in C_0^{p-var}([0,T], \mathbb{R}^d)$. We say that x is a p-rough path if there exists a sequence $x_n \in C_0^{1-var}([0,T], \mathbb{R}^d)$ such that $x_n \to x$ in p-variation and such that for every $\varepsilon > 0$, there exists $N \ge 0$ such that for $m, n \ge N$,*

$$\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|^{1/j}_{\frac{p}{j}-var,[0,T]} \le \varepsilon.$$

Our first goal is of course to relate the notion of geometric rough path to the notion of rough path.

Proposition 8.9. *Let $y \in C_0^{p-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ be a geometric p-rough path, then the projection of y onto $\mathbb{R}^d$ is a p-rough path.*

Proof. Let $y \in C_0^{p-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ be a geometric p-rough path and let us consider a sequence $y_n \in C_0^{1-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ that converges to y in the p-variation distance. Denote by x the projection of y onto $\mathbb{R}^d$ and by x_n the projection of y_n. From a previous theorem $y_n = S_{[p]}(x_n)$. It is clear that x_n converges to x in p-variation. So, we want to prove that for every $\varepsilon > 0$, there exists $N \geq 0$ such that for $m, n \geq N$,

$$\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \leq \varepsilon.$$

Let us now keep in mind that

$$d_{p-var;[s,t]}(y_n, y_m) = \left(\sup_{\Pi \in \mathcal{D}[s,t]} \sum_{k=0}^{n-1} d(y_n(t_k)^{-1} y_n(t_{k+1}), y_m(t_k)^{-1} y_m(t_{k+1}))^p \right)^{1/p}.$$

and consider the control

$$\omega(s,t) = \left(\frac{d_{p-var;[s,t]}(y_n, y_m)}{d_{p-var;[0,T]}(y_n, y_m)} \right)^p + \left(\frac{d_{p-var;[s,t]}(0, y_m)}{d_{p-var;[0,T]}(0, y_m)} \right)^p.$$

We have

$$\begin{aligned}
\left\| \int dx_n^{\otimes k} - \int dx_m^{\otimes k} \right\|_{\frac{p}{k}-var,[0,T]} &= \left(\sup_{\Pi \in \mathcal{D}[0,T]} \sum_{j=0}^{n-1} \left\| \int_{\Delta^k[t_j,t_{j+1}]} dx_n^{\otimes k} - \int_{\Delta^k[t_j,t_{j+1}]} dx_m^{\otimes k} \right\|^{p/k} \right)^{k/p} \\
&\leq \left(\sup_{0 \leq s \leq t \leq T} \frac{\left\| \int_{\Delta^k[s,t]} dx_n^{\otimes k} - \int_{\Delta^k[s,t]} dx_m^{\otimes k} \right\|}{\omega(s,t)^{k/p}} \right) \omega(0,T)^{k/p}
\end{aligned}$$

From the ball-box estimate, there is a constant C such that for $x, y \in \mathbb{G}_{[p]}(\mathbb{R}^d)$:

$$\|x - y\| \leq C \max\{d(x,y) \max\{1, d(0,x)^{N-1}\}, d(x,y)^N\}.$$

We deduce

$$\begin{aligned}
&\frac{\left\| \int_{\Delta^k[s,t]} dx_n^{\otimes k} - \int_{\Delta^k[s,t]} dx_m^{\otimes k} \right\|}{\omega(s,t)^{k/p}} \\
\leq &C \max \left\{ d_{p-var;[0,T]}(y_n, y_m) \max\{1, d_{p-var;[0,T]}(0, y_m)^{N-1}\}, d_{p-var;[0,T]}(y_n, y_m)^N \right\}
\end{aligned}$$

and thus

$$\left\| \int dx_n^{\otimes k} - \int dx_m^{\otimes k} \right\|_{\frac{p}{k}-var,[0,T]} \leq C' d_{p-var;[0,T]}(y_n, y_m)$$

This is the estimate we were looking for. □

Conversely, any p-rough path admits at least one lift as a geometric p-rough path.

Proposition 8.10. *Let $x \in C_0^{1-var}([0,T], \mathbb{R}^d)$ be a p-rough path. There exists a geometric p-rough path $y \in \mathbf{\Omega G}^p([0,T], \mathbb{R}^d)$ such that the projection of y onto $\mathbb{R}^d$ is x.*

Proof. Consider a sequence $x_n \in C_0^{1-var}([0,T], \mathbb{R}^d)$ such that $x_n \to x$ in p-variation and such that for every $\varepsilon > 0$, there exists $N \geq 0$ such that for $m, n \geq N$,

$$\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \leq \varepsilon.$$

We claim that $y_n S_{[p]}(x_n)$ is a sequence that converges in p-variation to some $y \in \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$ such that the projection of y onto $\mathbb{R}^d$ is x. Let us consider the control

$$\omega(s,t) = \left(\frac{\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[s,t]}^{1/j}}{\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j}} \right)^p + \left(\frac{d_{p-var;[s,t]}(0,y_m)}{d_{p-var;[0,T]}(0,y_m)} \right)^p.$$

We have

$$d_{p-var;[0,T]}(y_n,y_m) \le \left(\sup_{0\le s\le t\le T} \frac{d\left(y_n(s)^{-1}y_n(t), y_m(s)^{-1}y_m(t)\right)}{\omega(s,t)^{1/p}} \right) \omega(0,T)^{1/p}$$

and argue as above to get, thanks to the ball-box theorem, an estimate like

$$d_{p-var;[0,T]}(y_n,y_m) \le C \left(\sum_{j=1}^{[p]} \left\| \int dx_n^{\otimes j} - \int dx_m^{\otimes j} \right\|_{\frac{p}{j}-var,[0,T]}^{1/j} \right)^{1/N}$$

□

In general, we stress that there may be several geometric rough paths with the same projection onto $\mathbb{R}^d$. The following proposition is useful to prove that a given path is a geometric rough path.

Proposition 8.11. *If $q < p$, then $C_0^{q-var}([0,T],\mathbb{G}_{[p]}(\mathbb{R}^d)) \subset \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$.*

PROOF. As in Euclidean case, it is not difficult to prove that $x \in \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$ if and only if

$$\lim_{\delta\to 0} \sup_{\Pi\in\Delta[s,t],|\Pi|\le\delta} \sum_{k=0}^{n-1} d(x(t_k),x(t_{k+1}))^p = 0,$$

which is easy to check when $x \in C_0^{q-var}([0,T],\mathbb{G}_{[p]}(\mathbb{R}^d))$. □

If $y \in \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$, then as we just saw, the projection x of y onto $\mathbb{R}^d$ is a p-rough path and we can write

$$y(t) = 1 + \sum_{k=1}^{[p]} \int_{\Delta^k[0,t]} dx^{\otimes k}.$$

This is a convenient way to write geometric rough paths that we will often use in the sequel. For $N \ge [p]$ we can then define the lift of y in $\mathbf{\Omega G}^N([0,T],\mathbb{R}^d)$ as:

$$S_N(y)(t) = 1 + \sum_{k=1}^{N} \int_{\Delta^k[0,t]} dx^{\otimes k}.$$

The following result is then easy to prove by using the previous results.

Proposition 8.12. *Let $p \ge 1$ and $N \ge [p]$. There exist constants $C_1, C_2 > 0$ such that for every $y \in \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$,*

$$\|y\|_{p-var,[0,T]} \le \|S_N(y)\|_{p-var;[0,T]} \le C_2 \|y\|_{p-var,[0,T]}.$$

9. The Brownian motion as a rough path

It is now time to give a fundamental example of rough path: The Brownian motion. As we are going to see, a Brownian motion is a p-rough path for any $2 < p < 3$.

We first remind the following basic definition.

Definition 9.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A continuous d-dimensional process $(B_t)_{t\geq 0}$ is called a standard Brownian motion if it is a Gaussian process with mean function*

$$m(t) = 0$$

and covariance function

$$R(s,t) = \mathbb{E}(B_s \otimes B_t) = \min(s,t)\mathbf{I}_d.$$

For a Brownian motion $(B_t)_{t\geq 0}$, the following properties are easy to check:

(1) $B_0 = 0$ a.s.;
(2) For any $h \geq 0$, the process $(B_{t+h} - B_h)_{t\geq 0}$ is a standard Brownian motion;
(3) For any $t > s \geq 0$, the random variable $B_t - B_s$ is independent of the σ-algebra $\sigma(B_u, u \leq s)$.
(4) For every $c > 0$, the process $(B_{ct})_{t\geq 0}$ has the same law as the process $(\sqrt{c}B_t)_{t\geq 0}$.

An easy computation shows that for $n \geq 0$ and $0 \leq s \leq t$:

$$\mathbb{E}\left(\|B_t - B_s\|^{2n}\right) = \frac{(2n)!}{2^n n!}(t-s)^n.$$

Therefore, as a consequence of the Kolmogorov continuity theorem, for any $T \geq 0$ and $0\varepsilon < 1/2$, there exists a finite random variable $C_{T,\varepsilon}$ such that for $0 \leq s \leq t \leq T$,

$$\|B_t - B_s\| \leq C_{T,\varepsilon}|t-s|^{1/2-\varepsilon}.$$

We deduce in particular that for any $p > 2$, we have almost surely

$$\|B\|_{p-var,[0,T]} < +\infty.$$

We now prove that for $1 \leq p < 2$, we have almost surely

$$\|B\|_{p-var,[0,T]} = +\infty.$$

In the sequel, if

$$\Delta_n[0,t] = \{0 = t_0^n \leq t_1^n \leq ... \leq t_n^n = t\}$$

is a subdivision of the time interval $[0,t]$, we denote by

$$| \Delta_n[0,t] | = \max\{| t_{k+1}^n - t_k^n |, k = 0, ..., n-1\},$$

the mesh of this subdivision.

Proposition 9.2. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Let $t \geq 0$. For every sequence $\Delta_n[0,t]$ of subdivisions such that*

$$\lim_{n\to+\infty} | \Delta_n[0,t] | = 0,$$

the following convergence takes place in L^2 (and thus in probability),

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left\|B_{t_k^n}-B_{t_{k-1}^n}\right\|^2=t.$$

As a consequence, if $1\le p<2$, for every $T\ge 0$, almost surely,

$$\|B\|_{p-var,[0,T]}=+\infty.$$

Proof. We prove the result in dimension 1 and let the reader adapt it to the multi-dimensional setting. Let us denote

$$V_n=\sum_{k=1}^{n}\left(B_{t_k^n}-B_{t_{k-1}^n}\right)^2.$$

Thanks to the stationarity and the independence of Brownian increments, we have:

$$\begin{aligned}
\mathbb{E}\left((V_n-t)^2\right)&=\mathbb{E}\left(V_n^2\right)-2t\mathbb{E}\left(V_n\right)+t^2\\
&=\sum_{j,k=1}^{n}\mathbb{E}\left(\left(B_{t_j^n}-B_{t_{j-1}^n}\right)^2\left(B_{t_k^n}-B_{t_{k-1}^n}\right)^2\right)-t^2\\
&=\sum_{k=1}^{n}\mathbb{E}\left(\left(B_{t_j^n}-B_{t_{j-1}^n}\right)^4\right)+2\sum_{1\le j<k\le n}\mathbb{E}\left(\left(B_{t_j^n}-B_{t_{j-1}^n}\right)^2\left(B_{t_k^n}-B_{t_{k-1}^n}\right)^2\right)-t^2\\
&=\sum_{k=1}^{n}(t_k^n-t_{k-1}^n)^2\mathbb{E}\left(B_1^4\right)+2\sum_{1\le j<k\le n}(t_j^n-t_{j-1}^n)(t_k^n-t_{k-1}^n)-t^2\\
&=3\sum_{k=1}^{n}(t_j^n-t_{j-1}^n)^2+2\sum_{1\le j<k\le n}(t_j^n-t_{j-1}^n)(t_k^n-t_{k-1}^n)-t^2\\
&=2\sum_{k=1}^{n}(t_k^n-t_{k-1}^n)^2\\
&\le 2t\mid\Delta_n[0,t]\mid\to_{n\to+\infty}0.
\end{aligned}$$

Let us now prove that, as a consequence of this convergence, the paths of the process $(B_t)_{t\ge 0}$ almost surely have an infinite p-variation on the time interval $[0,t]$ if $1\le p<2$. Reasoning by absurd, let us assume that $\|B\|_{p-var,[0,t]}\le M$. From the above result, since the convergence in probability implies the existence of an almost surely convergent subsequence, we can find a sequence of subdivisions $\Delta_n[0,t]$ whose mesh tends to 0 and such that almost surely,

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(B_{t_k^n}-B_{t_{k-1}^n}\right)^2=t.$$

We get then

$$\sum_{k=1}^{n}\left(B_{t_k^n}-B_{t_{k-1}^n}\right)^2\le M^p\sup_{1\le k\le n}\mid B_{t_k^n}-B_{t_{k-1}^n}\mid^{2-p}\to_{n\to+\infty}0,$$

which is clearly absurd. □

Therefore only the case $p = 2$ is let open. It is actually possible to prove that:

Proposition 9.3. *For every $T \geq 0$, we have almost surely*

$$\|B\|_{2-var,[0,T]} = +\infty.$$

PROOF. See the book by Friz-Victoir page 381. □

In the previous Lecture we proved that Brownian motion paths almost surely have a bounded p-variation for every $p > 2$. In this lecture, we are going to prove that they even almost surely are p-rough paths for $2 < p < 3$. To prove this, we need to construct a geometric p rough path over the Brownian motion, that is we need to lift the Brownian motion to the free nilpotent Lie group of step 2, $\mathbb{G}_2(\mathbb{R}^d)$. In this process, we will have to define the iterated integrals $\int dB^{\otimes 2} = \int B \otimes dB$. This can be done by using the theory of stochastic integrals. Indeed, it is well known (and easy to prove !) that if

$$\Delta_n[0,t] = \{0 = t_0^n \leq t_1^n \leq ... \leq t_n^n = t\}$$

is a subdivision of the time interval $[0,t]$ whose mesh goes to 0, then the Riemann sums

$$\sum_{k=0}^{n-1} B_{t_k^n} \otimes (B_{t_{k+1}^n} - B_{t_k^n})$$

converge in probability to a random variable denoted $\int_0^t B_s \otimes dB_s$. We can then prove that the stochastic process $\int_0^t B_s \otimes dB_s$ admits a continuous version which is a martingale. With this integral of B against itself in hands, we can now proceed to construct a canonical geometric rough path over B.

Let $d \geq 2$ and denote $\mathcal{AS}_d$ the space of $d \times d$ skew-symmetric matrices. We can realize the group $\mathbb{G}_2(\mathbb{R}^d)$ in the following way

$$\mathbb{G}_2(\mathbb{R}^d) = (\mathbb{R}^d \times \mathcal{AS}_d, \circledast)$$

where $\circledast$ is the group law defined by

$$(\alpha_1, \omega_1) \circledast (\alpha_2, \omega_2) = (\alpha_1 + \alpha_2, \omega_1 + \omega_2 + \frac{1}{2}\alpha_1 \wedge \alpha_2).$$

Here we use the following notation; if $\alpha_1, \alpha_2 \in \mathbb{R}^d$, then $\alpha_1 \wedge \alpha_2$ denotes the skew-symmetric matrix $\left(\alpha_1^i \alpha_2^j - \alpha_1^j \alpha_2^i\right)_{i,j}$. Notice that the dilation writes

$$c \cdot (\alpha, \omega) = (c\alpha, c^2\omega). \tag{9.1}$$

Remark 9.4. *If $x : [0, +\infty) \to \mathbb{R}^2$ is a continuous path with bounded variation then for $0 < t_1 < t_2$ we denote*

$$\Delta_{[t_1,t_2]}x = \left(x_{t_2}^1 - x_{t_1}^1, x_{t_2}^2 - x_{t_1}^2, S_{[t_1,t_2]}x\right),$$

where $S_{[t_1,t_2]}x$ is the area swept out by the vector $\overrightarrow{x_{t_1}x_t}$ during the time interval $[t_1,t_2]$. Then, it is easily checked that for $0 < t_1 < t_2 < t_3$,

$$\Delta_{[t_1,t_3]}x = \Delta_{[t_1,t_2]}x \circledast \Delta_{[t_2,t_3]}x,$$

where $\circledast$ is precisely the law of $\mathbb{G}_2(\mathbb{R}^2)$, i.e. for (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \mathbb{R}^3$,

$$(x_1, y_1, z_1) \circledast (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2} (x_1 y_2 - x_2 y_1) \right).$$

We now are in position to give the fundamental definition.

Definition 9.5. *The process*

$$\mathbf{B}_t = \left(B_t, \frac{1}{2} \left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \right)_{1 \le i,j \le d} \right), \ t \ge 0.$$

is called the lift of the Brownian motion $(B_t)_{t \ge 0}$ in the group $\mathbb{G}_2(\mathbb{R}^d)$.

Interestingly, it turns out that the lift of a Brownian motion is a Markov process. Indeed, consider the vector fields

$$D_i(x) = \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{j<i} x^j \frac{\partial}{\partial x^{j,i}} - \frac{1}{2} \sum_{j>i} x^j \frac{\partial}{\partial x^{i,j}}, \ 1 \le i \le d,$$

defined on $\mathbb{R}^d \times \mathcal{AS}_d$. It is easy to check that:

(1) For $x \in \mathbb{R}^d \times \mathcal{AS}_d$,
$$[D_i, D_j](x) = \frac{\partial}{\partial x^{i,j}}, \ 1 \le i < j \le d;$$
(2) For $x \in \mathbb{R}^d \times \mathcal{AS}_d$,
$$[[D_i, D_j], D_k](x) = 0, \ 1 \le i, j, k \le d;$$
(3) The vector fields
$$(D_i, [D_j, D_k])_{1 \le i \le d, 1 \le j < k \le d}$$
are invariant with respect to the left action of $\mathbb{G}_2(\mathbb{R}^d)$ on itself and form a basis of the Lie algebra $\mathfrak{g}_2(\mathbb{R}^d)$ of $\mathbb{G}_2(\mathbb{R}^d)$.

The process $(\mathbf{B}_t)_{t \ge 0}$ solves the stochastic differential equation

$$d\mathbf{B}_t = \sum_{i=1}^d D_i(\mathbf{B}_t) \circ dB_s^i, \ 0 \le t \le T.$$

and as such, is a diffusion process in $\mathbb{R}^d \times \mathcal{AS}_d$ whose generator is the subelliptic diffusion operator given by

$$\frac{1}{2} \sum_{i=1}^d \frac{\partial^2}{\partial (x^i)^2} + \frac{1}{2} \sum_{i<j} \left(x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \right) \frac{\partial}{\partial x^{i,j}} + \frac{1}{8} \sum_{i<j} ((x^i)^2 + (x^j)^2) \frac{\partial^2}{\partial (x^{i,j})^2}.$$

Finally, also observe that we have the following scaling property, for every $c > 0$,

$$(\mathbf{B}_{ct})_{t \ge 0} =^{\text{law}} \left(\sqrt{c} \cdot \mathbf{B}_t \right)_{t \ge 0}.$$

Before we turn to the fundamental result of this Lecture, we need the following result which is known as the Garsia-Rodemich-Rumsey inequality (see the proof page 573 in the book by Friz-Victoir):

Lemma 9.6. *Let (X, d) be a metric space and $x : [0, T] \to E$ be a continuous path. Let $q > 1$ and $\alpha \in (1/q, 1)$. There exists a constant $C = C(\alpha, q)$ such that:*

$$d(x(s), x(t))^q \leq C|t-s|^{\alpha q - 1} \int_{[s,t]^2} \frac{d(x(u), x(v))^q}{|u-v|^{1+\alpha q}} du dv.$$

Theorem 9.7. *The paths of $(\mathbf{B}_t)_{t\geq 0}$ are almost surely geometric p-rough paths for $2 < p < 3$. As a consequence, the Brownian motion paths almost surely are p-rough paths for $2 < p < 3$. Let $q > 1$.*

Proof. We know that if $q < p$, then $C_0^{q-var}([0, T], \mathbb{G}_{[p]}(\mathbb{R}^d)) \subset \mathbf{\Omega G}^p([0, T], \mathbb{R}^d)$. Therefore, we need to prove that for $2 < p < 3$, the paths of $(\mathbf{B}_t)_{t\geq 0}$ almost surely have bounded p-variation with respect to the Carnot-Carathéodory distance. From the scaling property of $(\mathbf{B}_t)_{t\geq 0}$ and of the Carnot-Carathéodory distance, we have in distribution

$$d(\mathbf{B}_s, \mathbf{B}_t) =^d \sqrt{t-s} d(0, \mathbf{B}_1).$$

Moreover, from the equivalence of homogeneous norms, we have

$$d(0, \mathbf{B}_1) \simeq \|B_1\| + \left\| \int_0^1 B \otimes dB \right\|^{1/2}.$$

It easily follows from that, that for every $q > 1$,

$$\mathbb{E}\left(\frac{d(\mathbf{B}_s, \mathbf{B}_t)^q}{(t-s)^{q/2}}\right) = \mathbb{E}\left(d(0, \mathbf{B}_1)^q\right) < +\infty.$$

Thus, from Fubini's theorem we obtain

$$\mathbb{E}\left(\int_{[0,T]^2} \frac{d(\mathbf{B}_u, \mathbf{B}_v)^q}{|u-v|^{q/2}} du dv\right) < +\infty.$$

The Garsia-Rodemich-Rumsey inequality implies then

$$d(\mathbf{B}_s, \mathbf{B}_t)^q \leq C|t-s|^{q/2-1} \int_{[0,T]^2} \frac{d(\mathbf{B}_u, \mathbf{B}_v)^q}{|u-v|^{q/2}} du dv.$$

Therefore, the paths of $(\mathbf{B}_t)_{t\geq 0}$ almost surely have bounded p-variation for $p > 2$. $\square$

CHAPTER 5

Rough differential equations

In this chapter, we now study differential equations.

1. Davie's estimate

In this Lecture, we prove one of the fundamental estimates of rough paths theory. This estimate is due to Davie. It provides a basic estimate for the solution of the differential equation

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s)$$

in terms of the p-variation of the lift of x in the free Carnot group of step $[p]$.

We first introduce the somehow minimal regularity requirement on the vector fields V_i's to study rough differential equations.

Definition 1.1. *A vector field V on $\mathbb{R}^n$ is called γ-Lipschitz if it is $[\gamma]$ times continuously differentiable and there exists a constant $M \geq 0$ such that the supremum norm of its kth derivatives $k = 0, \cdots, [\gamma]$ and the $\gamma - [\gamma]$ Hölder norm of its $[\gamma]$th derivative are bounded by M. The smallest M that satisfies the above condition is the γ-Lipschitz norm of V and will be denoted $\|V\|_{Lip^\gamma}$.*

The fundamental estimate by Davie is the following;

Theorem 1.2. *Let $\gamma > p \geq 1$. Assume that $V_1, \cdots, V_d$ are $(\gamma - 1)$-Lipschitz vector fields in $\mathbb{R}^n$. Let $x \in C^{1-var}([0,T], \mathbb{R}^d)$. Let y be the solution of the equation*

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s), \quad 0 \leq t \leq T.$$

There exists a constant C depending only on p and γ such that for every $0 \leq s < t \leq T$,

$$\|y\|_{p-var,[s,t]} \leq C \left(\|V\|_{Lip^{\gamma-1}} \|S_{[p]}(x)\|_{p-var,[s,t]} + \|V\|^p_{Lip^{\gamma-1}} \|S_{[p]}(x)\|^p_{p-var,[s,t]} \right),$$

where $S_{[p]}(x)$ is the lift of x in $\mathbb{G}_{[p]}(\mathbb{R}^d)$.

We start with two preliminary lemmas, the first one being interesting in itself.

Lemma 1.3. *Let $\gamma > 1$. Assume that $V_1, \cdots, V_d$ are $(\gamma - 1)$-Lipschitz vector fields in $\mathbb{R}^n$. Let $x \in C^{1-var}([s,t], \mathbb{R}^d)$. Let y be the solution of the equation*

$$y(v) = y(s) + \sum_{i=1}^{d} \int_s^v V_i(y(u))dx^i(u), \quad s \leq v \leq t.$$

There exists a constant C depending only on γ such that,

$$\left\| y(t) - y(s) - \sum_{k=1}^{[\gamma]} \sum_{i_1,\cdots,i_k \in \{1,\cdots,d\}} V_{i_1}\cdots V_{i_k}\mathbf{I}(y(s)) \int_{\Delta^k[s,t]} dx^{i_1,\cdots,i_k} \right\| \le C \left(\|V\|_{Lip^{\gamma-1}} \int_s^t \|dx(r)\| \right)^{\gamma},$$

where $\mathbf{I}$ is the identity map.

Proof. For notational simplicity, we denote $n = [\gamma]$. An iterative use of the change of variable formula leads to

$$y(t) - y(s) - \sum_{k=1}^{n} \sum_{i_1,\cdots,i_k \in \{1,\cdots,d\}} V_{i_1}\cdots V_{i_k}\mathbf{I}(y(s)) \int_{\Delta^k[s,t]} dx^{i_1,\cdots,i_k}$$

$$= \int_{s<r_1<\cdots<r_n<t} \sum_{i_1,\cdots,i_n \in \{1,\cdots,d\}} (V_{i_1}\cdots V_{i_n}\mathbf{I}(y(r_1)) - V_{i_1}\cdots V_{i_n}\mathbf{I}(y(s))) dx^{i_1}_{r_1}\cdots dx^{i_n}_{r_n}.$$

Since $V_1,\cdots,V_d$ are $(\gamma-1)$-Lipschitz, we deduce that

$$\|V_{i_1}\cdots V_{i_n}\mathbf{I}(y(r_1)) - V_{i_1}\cdots V_{i_n}\mathbf{I}(y(s))\| \le \|V\|^n_{\mathrm{Lip}^{\gamma-1}} \|y(r_1) - y(s)\|^{\gamma-n}.$$

Since,

$$\|y(r_1) - y(s)\| \le \|V\|_{\mathrm{Lip}^{\gamma-1}} \int_s^{r_1} \|dx_r\|,$$

we deduce that

$$\|V_{i_1}\cdots V_{i_n}\mathbf{I}(y(r_1)) - V_{i_1}\cdots V_{i_n}\mathbf{I}(y(s))\| \le \|V\|^{\gamma}_{\mathrm{Lip}^{\gamma-1}} \left(\int_s^t \|dx_r\| \right)^{\gamma-n}.$$

The result follows then easily by plugging this estimate into the integral

$$\int_{s<r_1<\cdots<r_n<t} (V_{i_1}\cdots V_{i_n}\mathbf{I}(y(r_1)) - V_{i_1}\cdots V_{i_n}\mathbf{I}(y(s))) dx^{i_1}_{r_1}\cdots dx^{i_n}_{r_n}.$$

□

The second lemma is an analogue of a result already used in previous lectures (Young-Loeve estimate, estimates on iterated integrals).

Lemma 1.4. *Let $\Gamma : \{0 \le s \le t \le T\} \to \mathbb{R}^n$. Let us assume that:*

(1) *There exists a control $\tilde{\omega}$ such that*

$$\lim_{r\to 0} \sup_{(s,t)\in\Gamma, \tilde{\omega}(s,t)\le r} \frac{\|\Gamma_{s,t}\|}{r} = 0;$$

(2) *There exists a control ω and $\theta > 1, \xi > 0, K \ge 0, \alpha > 0$ such that for $0 \le s \le t \le u \le T$,*

$$\|\Gamma_{s,u}\| \le \left(\|\Gamma_{s,t}\| + \|\Gamma_{t,u}\| + \xi\omega(s,u)^{\theta}\right) \exp(K\omega(s,t)^{\alpha}).$$

Then, for all $0 \le s < t \le T$,

$$\|\Gamma_{s,t}\| \le \frac{\xi}{1-2^{1-\theta}} \omega(s,t)^{\theta} \exp\left(\frac{2K}{1-2^{-\alpha}} \omega(s,u)^{\alpha} \right).$$

PROOF. For $\varepsilon > 0$, consider then the control

$$\omega_\varepsilon(s,t) = \omega(s,t) + \varepsilon\tilde{\omega}$$

Define now

$$\Psi(r) = \sup_{s,u,\omega_\varepsilon(s,u)\le r} \|\Gamma_{s,u}\|.$$

If s, u is such that $\omega_\varepsilon(s,u) \le r$, we can find a t such that $\omega_\varepsilon(s,t) \le \frac{1}{2}\omega_\varepsilon(s,u)$, $\omega_\varepsilon(t,u) \le \frac{1}{2}\omega_\varepsilon(s,u)$. Indeed, the continuity of ω_ε forces the existence of a t such that $\omega_\varepsilon(s,t) = \omega_\varepsilon(t,u)$. We obtain therefore

$$\|\Gamma_{s,u}\| \le \left(2\Psi(r/2) + \xi r^\theta\right)\exp(Kr^\alpha),$$

which implies by maximization,

$$\Psi(r) \le \left(2\Psi(r/2) + \xi r^\theta\right)\exp(Kr^\alpha).$$

We have $\lim_{r\to 0}\frac{\Psi(r)}{r} = 0$ and an iteration easily gives

$$\Psi(r) \le \frac{\xi}{1-2^{1-\theta}} r^\theta \exp\left(\frac{2K}{1-2^{-\alpha}} r^\alpha\right).$$

We deduce

$$\|\Gamma_{s,t}\| \le \frac{\xi}{1-2^{1-\theta}} \omega_\varepsilon(s,t)^\theta \exp\left(\frac{2K}{1-2^{-\alpha}} \omega_\varepsilon(s,u)^\alpha\right),$$

and the result follows by letting $\varepsilon \to 0$. □

We now turn to the proof of Davie's estimate. We follow the approach by Friz-Victoir who smartly use interpolations by geodesics in Carnot groups.

Theorem 1.5. *Let $\gamma > p \ge 1$. Assume that $V_1, \cdots, V_d$ are $(\gamma-1)$-Lipschitz vector fields in $\mathbb{R}^n$. Let $x \in C^{1-var}([0,T],\mathbb{R}^d)$. Let y be the solution of the equation*

$$y(t) = y(0) + \sum_{i=1}^d \int_0^t V_i(y(s))dx^i(s), \quad 0 \le t \le T.$$

There exists a constant C depending only on p and γ such that for every $0 \le s < t \le T$,

$$\|y\|_{p-var,[s,t]} \le C\left(\|V\|_{Lip^{\gamma-1}}\|S_{[p]}(x)\|_{p-var,[s,t]} + \|V\|^p_{Lip^{\gamma-1}}\|S_{[p]}(x)\|^p_{p-var,[s,t]}\right),$$

where $S_{[p]}(x)$ is the lift of x in $\mathbb{G}_{[p]}(\mathbb{R}^d)$.

PROOF. For $s < t$, we denote by $x^{s,t}$ a path in $C^{1-var}([s,t],\mathbb{R}^d)$ such that $S_{[\gamma]}(x^{s,t})(s) = S_{[\gamma]}(x)(s)$, $S_{[\gamma]}(x^{s,t})(t) = S_{[\gamma]}(x)(t)$ and $S_{[\gamma]}(x^{s,t})(u)$, $s \le u \le t$, is a geodesic for the Carnot-Carathéodory distance. We consider then $y^{s,t}$ to be the solution of the equation

$$y^{s,t}(u) = y(s) + \sum_{i=1}^d \int_s^u V_i(y^{s,t}(v))dx^i(v), \quad s \le u \le t.$$

We can readily observe that from the continuity of Lyons' lift:

$$\|x^{s,t}\|_{1-var,[s,t]} = d(S_{[\gamma]}(x)(s), S_{[\gamma]}(x)(t)) \le \|S_{[\gamma]}(x)\|_{p-var,[s,t]} \le K\|S_{[p]}(x)\|_{p-var,[s,t]}.$$

Let us now denote

$$\Gamma_{s,t} = (y(t) - y(s)) - (y^{s,t}(t) - y^{s,t}(s)).$$

For fixed $s \leq t \leq u$, we have then:

$$\Gamma_{s,u} - \Gamma_{s,t} - \Gamma_{t,u} = (y^{s,u}(s) - y^{s,u}(u)) - (y^{s,t}(s) - y^{s,t}(t)) - (y^{t,u}(t) - y^{t,u}(u)).$$

To estimate this quantity, we consider the path $y^{s,t,u}(v)$, $s \leq v \leq u$, that solves the ordinary differential equation driven by the concatenation of $x^{s,t}$ and $x^{t,u}$. We first estimate $y^{s,t,u}(u) - y^{s,u}(u)$ by observing that $y^{s,t,u}(u)$ and $y^{s,u}(u)$ have the same Taylor expansion up to order $[\gamma]$. Thus by using the lemma of the previous lecture and the triangle inequality, we easily get that:

$$\begin{aligned}\|y^{s,t,u}(u) - y^{s,u}(u)\| &\leq C_1 \|V\|^{\gamma}_{\mathrm{Lip}^{\gamma-1}} \left(\int_s^t \|dx^{s,t}(r)\| + \int_t^u \|dx^{t,u}(r)\| \right)^{\gamma} \\ &\leq C_2 \|V\|^{\gamma}_{\mathrm{Lip}^{\gamma-1}} \|S_{[p]}(x)\|^{\gamma}_{p-var,[s,u]}.\end{aligned}$$

We then estimate $(y^{s,t,u}(u) - y^{s,t,u}(s)) + (y^{s,t}(s) - y^{s,t}(t)) + (y^{t,u}(t) - y^{t,u}(u))$ by observing that $y^{s,t,u}(s) = y^{s,t}(s)$, $y^{s,t,u}(t) = y^{s,t}(t)$. Thus,

$$\begin{aligned}&(y^{s,t,u}(u) - y^{s,t,u}(s)) + (y^{s,t}(s) - y^{s,t}(t)) + (y^{t,u}(t) - y^{t,u}(u)) \\ =&(y^{s,t,u}(u) - y^{s,t,u}(t)) - (y^{t,u}(u) - y^{t,u}(t))\end{aligned}$$

This last term is estimated by using basic continuity estimates with respect to the initial condition which gives

$$\begin{aligned}&\|(y^{s,t,u}(u) - y^{s,t,u}(t)) - (y^{t,u}(u) - y^{t,u}(t))\| \\ \leq&\|y^{s,t,u}(t) - y^{t,u}(t)\| \|V\|_{\mathrm{Lip}^{\gamma-1}} \int_t^u \|dx^{t,u}(r)\| \exp\left(\|V\|_{\mathrm{Lip}^{\gamma-1}} \int_t^u \|dx^{t,u}(r)\| \right) \\ \leq&C_3 \|\Gamma_{s,t}\| \|V\|_{\mathrm{Lip}^{\gamma-1}} \|S_{[p]}(x)\|_{p-var,[t,u]} \exp\left(C_3 \|V\|_{\mathrm{Lip}^{\gamma-1}} \|S_{[p]}(x)\|_{p-var,[t,u]} \right)\end{aligned}$$

We conclude

$$\|\Gamma_{s,u} - \Gamma_{s,t} - \Gamma_{t,u}\| \leq C_2 \omega(s,u)^{\gamma/p} + C_3 \|\Gamma_{s,t}\| \omega(t,u)^{1/p} \exp\left(C_3 \omega(t,u)^{1/p} \right),$$

where

$$\omega(s,t) = \left(\|V\|_{\mathrm{Lip}^{\gamma-1}} \|S_{[p]}(x)\|_{p-var,[s,t]} \right)^p.$$

The basic inequality $1 + xe^x \leq e^{2x}$ combined with the triangle inequality gives:

$$\begin{aligned}\|\Gamma_{s,u}\| &\leq \|\Gamma_{t,u}\| + \|\Gamma_{s,t}\| \exp\left(2C_3 \omega(s,u)^{1/p} \right) + C_2 \omega(s,u)^{\gamma/p} \\ &\leq \left(\|\Gamma_{t,u}\| + \|\Gamma_{s,t}\| + C_2 \omega(s,u)^{\gamma/p} \right) \exp\left(2C_3 \omega(s,u)^{1/p} \right).\end{aligned}$$

We are now in position to apply the lemma of the previous lecture (we let the reader check that the assumptions are satisfied). We deduce then

$$\|\Gamma_{s,t}\| \leq C_4 \omega(s,t)^{\gamma/p} \exp\left(C_4 \omega(s,t)^{1/p} \right).$$

We now keep in mind that

$$\Gamma_{s,t} = (y(t) - y(s)) - (y^{s,t}(t) - y^{s,t}(s)),$$

and $y^{s,t}(t) - y^{s,t}(s)$ can be estimated by using basic estimates on differential equations:

$$\begin{aligned}\|y^{s,t}(t) - y^{s,t}(s)\| &\le C_5 \|V\|_{\mathrm{Lip}^{\gamma-1}} \int_s^t \|dx^{s,t}(u)\| \\ &\le C_6 \omega(s,t)^{1/p}.\end{aligned}$$

From the triangle inequality, we conclude then:

$$\|y(s) - y(t)\| \le C_6 \omega(s,t)^{1/p} + C_4 \omega(s,t)^{\gamma/p} \exp\left(C_4 \omega(s,t)^{1/p}\right),$$

In particular we have for s,t such that $\omega(s,t) \le 1$,

$$\|y(s) - y(t)\| \le C_7 \omega(s,t)^{1/p}.$$

This easily gives the required estimate (see Proposition 5.10 in the book by Friz-Victoir).

□

We can remark that the proof actually also provided the following estimate which is interesting in itself:

Proposition 1.6. *Let $\gamma > p \ge 1$. Assume that $V_1, \cdots, V_d$ are $(\gamma - 1)$-Lipschitz vector fields in $\mathbb{R}^n$. Let $x \in C^{1-var}([0,T], \mathbb{R}^d)$. Let y be the solution of the equation*

$$y(t) = y(0) + \sum_{i=1}^d \int_0^t V_i(y(s)) dx^i(s), \quad 0 \le t \le T.$$

There exists a constant C depending only on p and γ such that for every $0 \le s < t \le T$,

$$\left\| y(t) - y(s) - \sum_{k=1}^{[\gamma]} \sum_{i_1, \cdots, i_k \in \{1, \cdots, d\}} V_{i_1} \cdots V_{i_k} \mathbf{I}(y(s)) \int_{\Delta^k[s,t]} dx^{i_1, \cdots, i_k} \right\| \le C \|V\|^{\gamma}_{Lip^{\gamma-1}} \|S_{[p]}(x)\|^{\gamma}_{p-var,[s,t]}$$

2. The Lyons' continuity theorem

We are now ready to state the main theorem of rough paths theory: the continuity of solutions of differential equations with respect to the driving path.

Theorem 2.1. *Let $\gamma > p \ge 1$. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let $x_1, x_2 \in C^{1-var}([0,T], \mathbb{R}^d)$ such that*

$$\|S_{[p]}(x_1)\|^p_{p-var,[0,T]} + \|S_{[p]}(x_2)\|^p_{p-var,[0,T]} \le K$$

with $K \ge 0$. Let y_1, y_2 be the solutions of the equations

$$y_i(t) = y(0) + \sum_{j=1}^d \int_0^t V_j(y_i(s)) dx_i^j(s), \quad 0 \le t \le T, \quad i = 1,2$$

There exists a constant C depending only on p, γ and K such that for $0 \le s \le t \le T$,

$$\|(y_2(t) - y_2(s)) - (y_1(t) - y_1(s))\| \le C \|V\|_{Lip^\gamma} e^{C\|V\|^p_{Lip^\gamma}} d_{p-var,[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2)) \omega(s,t)^{1/p},$$

where ω is the control

$$\omega(s,t) = \left(\frac{d_{p-var,[s,t]}(S_{[p]}(x_1), S_{[p]}(x_2))}{d_{p-var;[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2))} \right)^p + \left(\frac{\|S_{[p]}(x_1)\|_{p-var,[s,t]}}{\|S_{[p]}(x_1)\|_{p-var,[0,T]}} \right)^p + \left(\frac{\|S_{[p]}(x_2)\|_{p-var,[s,t]}}{\|S_{[p]}(x_2)\|_{p-var,[0,T]}} \right)^p.$$

The proof will take us some time and will be preceeded by several lemmas. We can however already give the following important corollaries:

Corollary 2.2 (Lyon's continuity theorem). *Let $\gamma > p \geq 1$. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let $x_1, x_2 \in C^{1-var}([0,T], \mathbb{R}^d)$ such that*

$$\| S_{[p]}(x_1) \|^p_{p-var,[0,T]} + \| S_{[p]}(x_2) \|^p_{p-var,[0,T]} \leq K$$

with $K \geq 0$. Let y_1, y_2 be the solutions of the equations

$$y_i(t) = y(0) + \sum_{j=1}^d \int_0^t V_j(y_i(s)) dx_i^j(s), \quad 0 \leq t \leq T, \quad i = 1, 2$$

There exists a constant C depending only on p, γ and K such that for $0 \leq s \leq t \leq T$,

$$\| y_2 - y_1 \|_{p-var,[0,T]} \leq C \| V \|_{Lip^\gamma} e^{C \| V \|^p_{Lip^\gamma}} d_{p-var,[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2)).$$

This continuity statement immediately suggests the following basic definition for solutions of differential equation driven by p-rough paths.

Theorem 2.3. *Let $p \geq 1$. Let $\mathbf{x} \in \mathbf{\Omega G}^p([0,T], \mathbb{R}^d)$ be a geometric p-rough path over the p-rough path x. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$ with $\gamma > p$. If $\mathbf{x}_n \in C^{1-var}([0,T], \mathbb{G}_{[p]}(\mathbb{R}^d))$ is a sequence that converges to $\mathbf{x}$ in p-variation, then the solution of the equation*

$$y_n(t) = y(0) + \sum_{j=1}^d \int_0^t V_j(y_n(s)) dx_n^j(s), \quad 0 \leq t \leq T,$$

converges in p-variation to some $y \in C^{p-var}([0,T], \mathbb{R}^d)$ that does not depend on the choice of the approximating sequence $\mathbf{x}_n$ and that we call a solution of the rough differential equation:

$$y(t) = y(0) + \sum_{j=1}^d \int_0^t V_j(y(s)) dx^j(s), \quad 0 \leq t \leq T.$$

The following propositions are easily obtained by a limiting argument:

Proposition 2.4 (Davie's estimate for rough differential equations). *Let $\gamma > p \geq 1$. Let $\mathbf{x} \in \mathbf{\Omega G}^p([0,T], \mathbb{R}^d)$ be a geometric p-rough path over the p-rough path x. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let y be the solution of the rough differential equation*

$$y(t) = y(0) + \sum_{i=1}^d \int_0^t V_i(y(s)) dx^i(s), \quad 0 \leq t \leq T.$$

There exists a constant C depending only on p and γ such that for every $0 \leq s < t \leq T$,

$$\| y \|_{p-var,[s,t]} \leq C \left(\| V \|_{Lip^{\gamma-1}} \| \mathbf{x} \|_{p-var,[s,t]} + \| V \|^p_{Lip^{\gamma-1}} \| \mathbf{x} \|^p_{p-var,[s,t]} \right).$$

Proposition 2.5. *Let $\gamma > p \geq 1$. Let $\mathbf{x} \in \mathbf{\Omega G}^p([0,T],\mathbb{R}^d)$ be a geometric p-rough path over the p-rough path x. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let y be the solution of the rough differential equation*

$$y(t) = y(0) + \sum_{i=1}^{d} \int_0^t V_i(y(s))dx^i(s), \quad 0 \leq t \leq T.$$

There exists a constant C depending only on p and γ such that for every $0 \leq s < t \leq T$,

$$\left\| y(t) - y(s) - \sum_{k=1}^{[\gamma]} \sum_{i_1,\cdots,i_k \in \{1,\cdots,d\}} V_{i_1}\cdots V_{i_k}\mathbf{I}(y(s)) \int_{\Delta^k[s,t]} dx^{i_1,\cdots,i_k} \right\| \leq C\|V\|^{\gamma}_{Lip^{\gamma-1}} \|\mathbf{x}\|^{\gamma}_{p-var,[s,t]}$$

We now turn to the proof of the continuity theorem. We start with several lemmas, which are not difficult but a little technical. The first one is geometrically very intuitive.

Lemma 2.6. *Let $g_1, g_2 \in \mathbb{G}_N(\mathbb{R}^d)$ such that $d(g_1,g_2) \leq \varepsilon$ with $\varepsilon > 0$ and $d(0,g_1), d(0,g_2) \leq K$ with $K \geq 0$. Then, there exists $x_1, x_2 \in C^{1-var}([0,1],\mathbb{R}^d)$ and a constant $C = C(N,K)$ such that $S_N(x)(1) = S_N(x)(0)g_i$, $i = 1,2$ and*

$$\|x_1\|_{1-var,[0,1]} + \|x_2\|_{1-var,[0,1]} \leq C$$

and

$$\|x_1 - x_2\|_{1-var,[0,1]} \leq \varepsilon C.$$

PROOF. See the book by Friz-Victoir, page 161. □

The next ingredient is the following estimate.

Lemma 2.7. *Let $\gamma \geq 1$. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let $x_1, \tilde{x}_1, x_2, \tilde{x}_2 \in C^{1-var}([0,T],\mathbb{R}^d)$ such that*

$$S_{[\gamma]}(x_1)(T) = S_{[\gamma]}(\tilde{x}_1)(T), \quad S_{[\gamma]}(x_2)(T) = S_{[\gamma]}(\tilde{x}_2)(T).$$

Let $y_1, y_2, \tilde{y}_1, \tilde{y}_2$ be the solutions of the equations

$$y_i(t) = y_i(0) + \sum_{j=1}^{d} \int_0^t V_j(y_i(s))dx_i^j(s), \quad 0 \leq t \leq T, \quad i = 1,2$$

and

$$\tilde{y}_i(t) = y_i(0) + \sum_{j=1}^{d} \int_0^t V_j(\tilde{y}_i(s))d\tilde{x}_i^j(s), \quad 0 \leq t \leq T, \quad i = 1,2.$$

If

$$\|x_1\|_{1-var,[0,T]} + \|\tilde{x}_1\|_{1-var,[0,T]} + \|x_2\|_{1-var,[0,T]} + \|\tilde{x}_2\|_{1-var,[0,T]} \leq K$$

and

$$\|x_1 - x_2\|_{1-var,[0,T]} + \|\tilde{x}_1 - \tilde{x}_2\|_{1-var,[0,T]} \leq M,$$

then, for some constant depending only on γ,

$$\begin{aligned} &\|(y_1(T) - \tilde{y}_1(T)) - (y_2(T) - \tilde{y}_2(T))\| \\ \leq &C\|y_1(0) - y_2(0)\|(\|V\|_{Lip^\gamma}K)^\gamma e^{C\|V\|_{Lip^\gamma}K} + CM\|V\|_{Lip^\gamma}(\|V\|_{Lip^\gamma}K)^\gamma e^{C\|V\|_{Lip^\gamma}K} \end{aligned}$$

PROOF. Let us first observe that it is enough to prove the result when $\tilde{x}_1 = \tilde{x}_2 = 0$. Indeed, suppose that we can prove the result in that case. Define then the path z to be the concatenation of $\tilde{x}_1(T-\cdot)$ and $x_1(\cdot)$ reparametrized so that $z : [0,T] \to \mathbb{R}^d$. It is seen that the solution of the equation

$$w(t) = \tilde{y}_1(T) + \sum_{j=1}^d \int_0^t V_j(w(s)) dz_i^j(s), \quad 0 \le t \le T$$

satisfies

$$w(T) - w(0) = y_1(T) - \tilde{y}_1(T).$$

We thus assume that $\tilde{x}_1 = \tilde{x}_2 = 0$. In that case, from the assumption, we have

$$S_{[\gamma]}(x_1)(T) = 1, \quad S_{[\gamma]}(x_2)(T) = 1.$$

Taylor's expansion gives then, with $n = [\gamma]$,

$$y_1(T) - y_1(0) = \int_{s \le r_1 \le \cdots \le r_n \le t} \sum_{i_1,\cdots,i_n \in \{1,\cdots,d\}} (V_{i_1} \cdots V_{i_n} \mathbf{I}(y_1(r_1)) - V_{i_1} \cdots V_{i_n} \mathbf{I}(y_1(s))) dx_{1,r_1}^{i_1} \cdots dx_{1,r_n}^{i_n}.$$

and similarly

$$y_2(T) - y_2(0) = \int_{s \le r_1 \le \cdots \le r_n \le t} \sum_{i_1,\cdots,i_n \in \{1,\cdots,d\}} (V_{i_1} \cdots V_{i_n} \mathbf{I}(y_2(r_1)) - V_{i_1} \cdots V_{i_n} \mathbf{I}(y_2(s))) dx_{2,r_1}^{i_1} \cdots dx_{2,r_n}^{i_n}.$$

The result is then easily obtained by using classical estimates for Riemann-Stieltjes integrals (details can be found page 230 in the book by Friz-Victoir). □

Finally, the last lemma is an easy consequence of Gronwall's lemma

Lemma 2.8. *Let $\gamma \ge 1$. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let $x_1, x_2 \in C^{1-var}([0,T], \mathbb{R}^d)$. Let $y_1, y_2, \tilde{y}_1, \tilde{y}_2$ be the solutions of the equations*

$$y_i(t) = y_i(0) + \sum_{j=1}^d \int_0^t V_j(y_i(s)) dx_i^j(s), \quad 0 \le t \le T, \quad i = 1,2$$

and

$$\tilde{y}_i(t) = \tilde{y}_i(0) + \sum_{j=1}^d \int_0^t V_j(\tilde{y}_i(s)) dx_i^j(s), \quad 0 \le t \le T, \quad i = 1,2.$$

If

$$\|x_1\|_{1-var,[0,T]} + \|x_2\|_{1-var,[0,T]} \le K$$

and

$$\|x_1 - x_2\|_{1-var,[0,T]} \le M,$$

then, for some constant depending only on γ,

$$\begin{aligned} &\|(y_1(T) - y_1(0)) - (\tilde{y}_1(T) - \tilde{y}_1(0)) - (y_2(T) - y_2(0)) + (\tilde{y}_2(T) - \tilde{y}_2(0))\| \\ \le & C\|V\|_{Lip^\gamma} K e^{C\|V\|_{Lip^\gamma} K} \|y_1(0) - \tilde{y}_1(0) - y_2(0) + \tilde{y}_2(0)\| + C\|V\|_{Lip^\gamma} M e^{C\|V\|_{Lip^\gamma} K} \\ & + C\|V\|_{Lip^\gamma} K e^{C\|V\|_{Lip^\gamma} K} (\|y_1(0) - \tilde{y}_1(0)\| + \|y_2(0) - \tilde{y}_2(0)\|)^{\min(2,\gamma)-1} \left(\|\tilde{y}^1(0) - \tilde{y}^2(0)\| + \|V\|_{Lip^\gamma} K \right) \end{aligned}$$

Proof. See the book by Friz-Victoir page 232. $\square$

We now turn to the proof of the Lyons' continuity theorem.

Theorem 2.9. *Let $\gamma > p \geq 1$. Assume that $V_1, \cdots, V_d$ are γ-Lipschitz vector fields in $\mathbb{R}^n$. Let $x_1, x_2 \in C^{1-var}([0,T], \mathbb{R}^d)$ such that*

$$\|S_{[p]}(x_1)\|^p_{p-var,[0,T]} + \|S_{[p]}(x_2)\|^p_{p-var,[0,T]} \leq K$$

with $K \geq 0$. Let y_1, y_2 be the solutions of the equations

$$y_i(t) = y(0) + \sum_{j=1}^{d} \int_0^t V_j(y_i(s)) dx_i^j(s), \quad 0 \leq t \leq T, \quad i = 1, 2$$

There exists a constant C depending only on p, γ and K such that for $0 \leq s \leq t \leq T$,

$$\|(y_2(t)-y_2(s))-(y_1(t)-y_1(s))\| \leq C\|V\|_{Lip^\gamma} e^{C\|V\|^p_{Lip^\gamma}} d_{p-var,[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2))\omega(s,t)^{1/p},$$

where ω is the control

$$\omega(s,t) = \left(\frac{d_{p-var,[s,t]}(S_{[p]}(x_1), S_{[p]}(x_2))}{d_{p-var;[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2))}\right)^p + \left(\frac{\|S_{[p]}(x_1)\|_{p-var,[s,t]}}{\|S_{[p]}(x_1)\|_{p-var,[0,T]}}\right)^p + \left(\frac{\|S_{[p]}(x_2)\|_{p-var,[s,t]}}{\|S_{[p]}(x_2)\|_{p-var,[0,T]}}\right)^p.$$

Proof. We may assume $p < \gamma < [p] + 1$, and for conciseness of notations, we set $\varepsilon = d_{p-var,[0,T]}(S_{[p]}(x_1), S_{[p]}(x_2))$. Let

$$g_i = \Delta_{\frac{1}{\omega(s,t)^{1/p}}}(S_{[p]}(x_i)(s)^{-1} S_{[p]}(x_i)(t)), \quad i = 1, 2.$$

We have,

$$\begin{aligned} d(g_1, g_2) &= \frac{1}{\omega(s,t)^{1/p}} d(S_{[p]}(x_1)(s)^{-1} S_{[p]}(x_1)(t), S_{[p]}(x_2)(s)^{-1} S_{[p]}(x_2)(t)) \\ &\leq \frac{1}{\omega(s,t)^{1/p}} d_{p-var,[s,t]}(S_{[p]}(x_1), S_{[p]}(x_2)) \\ &\leq \varepsilon \end{aligned}$$

and, in the same way,

$$\begin{aligned} d(0, g_i) &= \frac{1}{\omega(s,t)^{1/p}} d(S_{[p]}(x_i)(s), S_{[p]}(x_i)(t)) \\ &= \frac{1}{\omega(s,t)^{1/p}} \|S_{[p]}(x_i)\|_{p-var,[s,t]} \leq K. \end{aligned}$$

Therefore, there exists $x_1^{s,t}, x_2^{s,t} \in C^{1-var}([s,t], \mathbb{R}^d)$ and a constant $C_1 = C_1([p], K)$ such that

$$S_{[p]}(x_i^{s,t})(s)^{-1} S_{[p]}(x_i^{s,t})(t) = S_{[p]}(x_i)(s)^{-1} S_{[p]}(x_i)(t), i = 1, 2$$

and

$$\|x_1^{s,t}\|_{1-var,[s,t]} + \|x_2^{s,t}\|_{1-var,[s,t]} \leq C_1 \omega(s,t)^{1/p}$$

and

$$\|x_1^{s,t} - x_2^{s,t}\|_{1-var,[s,t]} \leq \varepsilon C_1 \omega(s,t)^{1/p}.$$

We define then $x_i^{s,t,u}$ as the concatenation of $x_i^{s,t}$ and $x_i^{t,u}$. As in the proof of Davie's lemma, we denote by $y_i^{s,t}$ the solution of the equation

$$y_i^{s,t}(r) = y_i(s) + \sum_{j=1}^{d} \int_s^r V_j(y_i^{s,t}(v))dx_i^j(v), \quad s \le r \le t, \quad i = 1, 2$$

and consider the functionals

$$\Gamma^i_{s,t} = (y_i(t) - y_i(s)) - (y_i^{s,t}(t) - y_i^{s,t}(s)) = y_i(t) - y_i^{s,t}(t),$$

and

$$\bar{\Gamma}_{s,t} = \Gamma^1_{s,t} - \Gamma^2_{s,t}$$

From the proof of Davie's estimate, it is seen that

$$\|\Gamma^i_{s,t}\| \le \frac{1}{2} C_2 \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p} \right)^{[p]+1},$$

and thus

$$\|\bar{\Gamma}_{s,t}\| \le C_2 \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p} \right)^{[p]+1}.$$

On the other hand, by estimating

$$\bar{\Gamma}_{s,u} - \bar{\Gamma}_{s,t} - \bar{\Gamma}_{t,u},$$

as in the proof of Davie's lemma, that is by inserting $y_i^{s,t,u}$ which is the solution of the equation driven by the concatenation of $x_i^{s,t}$ and $x_i^{t,u}$, and then by using the two lemmas of the previous lecture, we obtain the estimate

$$\begin{aligned} \|\bar{\Gamma}_{s,u}\| &\le \|\bar{\Gamma}_{s,t}\| e^{C_3 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p}} + \|\bar{\Gamma}_{t,u}\| + C_3(\|y_1 - y_2\|_{\infty,[s,t]} + \varepsilon) \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p} \right)^\gamma e^{C_3 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p}} \\ &\le \left(\|\bar{\Gamma}_{s,t}\| + \|\bar{\Gamma}_{t,u}\| + C_3(\|y_1 - y_2\|_{\infty,[s,t]} + \varepsilon) \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p} \right)^\gamma \right) e^{C_3 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p}}. \end{aligned}$$

It remains to bound $\|y_1 - y_2\|_{\infty,[s,t]}$. For this let us observe that

$$\|(y_1(t) - y_2(t)) - (y_1(s) - y_2(s)) - \bar{\Gamma}_{s,t}\| = \|(y_1^{s,t}(t) - y_2^{s,t}(t)) - (y_1^{s,t}(s) - y_2^{s,t}(s))\|.$$

$\|(y_1^{s,t}(t) - y_2^{s,t}(t)) - (y_1^{s,t}(s) - y_2^{s,t}(s))\|$ can then be estimated by using classical estimates on differential equations driven by bounded variation paths. This gives,

$$\|(y_1^{s,t}(t) - y_2^{s,t}(t)) - (y_1^{s,t}(s) - y_2^{s,t}(s))\| \le C_4 \left(\|y_1(s) - y_2(s)\| + \varepsilon \right) \|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p} e^{C_4 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p}}.$$

By denoting $z = y_1 - y_2$, we can summarize the two above estimates as follows:

$$\|\bar{\Gamma}_{s,u}\| \le \left(\|\bar{\Gamma}_{s,t}\| + \|\bar{\Gamma}_{t,u}\| + C_3(\|z\|_{\infty,[s,t]} + \varepsilon) \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p} \right)^\gamma \right) e^{C_3 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,u)^{1/p}}$$

and

$$\|z(t) - z(s) - \bar{\Gamma}_{s,t}\| \le C_4 \left(\|z\|_{\infty,[0,s]} + \varepsilon \right) \|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p} e^{C_4 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p}}.$$

From a lemma already used in the proof of Davie's estimate, the first estimate implies

$$\|\bar{\Gamma}_{s,t}\| \le C_5 \left(\varepsilon + \|z\|_{\infty,[0,t]} \right) \left(\|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p} \right)^\gamma e^{C_5 \|V\|_{\mathrm{Lip}^\gamma} \omega(s,t)^{1/p}}.$$

Using now the second estimate we obtain that for any interval $[a,b]$ included in $[0,T]$,

$$\sup_{s,t\in[a,b]} \|z(t)-z(s)\| \le C_6(\varepsilon + \|z\|_{\infty,[0,b]})\|V\|_{\mathrm{Lip}^\gamma}\omega(a,b)^{1/p}e^{C_6\|V\|_{\mathrm{Lip}^\gamma}\omega(a,b)^{1/p}}.$$

Using the fact that $z(0)=0$ and picking a subdivision $0=\tau_0\le\tau_1\le\cdots\le\tau_N\le T$ such that

$$C_6\|V\|_{\mathrm{Lip}^\gamma}e^{C_6\|V\|_{\mathrm{Lip}^\gamma}}\omega(\tau_i,\tau_{i+1})^{1/p}\le 1/2$$

we see that it implies

$$\|z\|_{\infty,[0,T]}\le C_7\varepsilon e^{C_7\|V\|^p_{\mathrm{Lip}^\gamma}}.$$

Coming back to the estimate

$$\sup_{s,t\in[a,b]} \|z(t)-z(s)\| \le C_6(\varepsilon + \|z\|_{\infty,[0,b]})\|V\|_{\mathrm{Lip}^\gamma}\omega(a,b)^{1/p}e^{C_6\|V\|_{\mathrm{Lip}^\gamma}\omega(a,b)^{1/p}},$$

concludes the proof. $\square$

CHAPTER 6

Applications to stochastic differential equations

This chapter is devoted to simple but spectacular applications of rough paths theory to stochastic differential equations.

1. Approximation of the Brownian rough path

Our goal in the next two lectures will be to prove that rough differential equations driven a Brownian motion seen as a p-rough path, $2 < p < 3$ are nothing else but stochastic differential equations understood in the Stratonovitch sense. The proof of this fact requires an explicit approximation of the Brownian rough path in the rough path topology which is interesting in itself.

Let $(B_t)_{t\geq 0}$ be a n-dimensional Brownian motion and let us denote by

$$\mathbf{B}_t = \left(B_t, \frac{1}{2} \left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \right)_{1\leq i<j\leq n} \right)$$

its lift in the free Carnot group of step 2 over $\mathbb{R}^d$.

Let us work on a fixed interval $[0,T]$ and consider a sequence D_n of subdivisions of $[0,T]$ such that $D_{n+1} \subset D_n$ and whose mesh goes to 0 when $n \to +\infty$. An example is given by the sequence of dyadic subdivisions. The family $\mathcal{F}_n = \sigma(B_t, t \in D_n)$ is then a filtration, that is an increasing family of σ-fields. We denote by B^n the piecewise linear process which is obtained from B by interpolation along the subdivision D_n, that is for $t_i^n \leq t \leq t_{i+1}^n$,

$$B_t^n = \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} B_{t_i} + \frac{t - t_i^n}{t_{i+1}^n - t_i^n} B_{t_{i+1}}.$$

The corresponding lifted process is then

$$\mathbf{B}_t^n = \left(B_t^n, \frac{1}{2} \left(\int_0^t B_s^{n,i} dB_s^{n,j} - B_s^{n,j} dB_s^{n,i} \right)_{1\leq i<j\leq n} \right).$$

The main result of the lecture is the following:

Theorem 1.1. *Let $2 < p < 3$. When $n \to +\infty$, almost surely, $d_{p-var,[0,T]}(\mathbf{B}^n, \mathbf{B}) \to 0$.*

We split the proof in two lemmas.

Lemma 1.2. *Let $t \in [0,T]$. When $n \to +\infty$, almost surely, $d(\mathbf{B}_t^n, \mathbf{B}_t) \to 0$.*

PROOF. We first observe that, due to the Markov property of Brownian motion, we have for $t_i^n \le t \le t_{i+1}^n$,

$$\mathbb{E}\left(B_t \mid \mathcal{F}_n\right) = \mathbb{E}\left(B_t \mid B_{t_i^n}, B_{t_i^{n+1}}\right).$$

It is then an easy exercise to check that

$$\mathbb{E}\left(B_t \mid B_{t_i^n}, B_{t_i^{n+1}}\right) = \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} B_{t_i} + \frac{t - t_i^n}{t_{i+1}^n - t_i^n} B_{t_{i+1}} = B_t^n.$$

As a conclusion, we get

$$\mathbb{E}\left(B_t \mid \mathcal{F}_n\right) = B_t^n.$$

It immediately follows that $B_t^n \to B_t$ when $n \to +\infty$. In the same way, we have

$$\mathbb{E}\left(\int_0^t B_s^i dB_s^j - B_s^j dB_s^i \mid \mathcal{F}_n\right) = \int_0^t B_s^{n,i} dB_s^{n,j} - B_s^{n,j} dB_s^{n,i}.$$

Indeed, for $0 < t < T$ and ε small enough, we have by independence of B^i and B^j,

$$\mathbb{E}\left(B_t^i(B_{t+\varepsilon}^j - B_t^j) \mid \mathcal{F}_n\right) = \mathbb{E}\left(B_t^i \mid \mathcal{F}_n\right)\mathbb{E}\left(B_{t+\varepsilon}^j - B_t^j) \mid \mathcal{F}_n\right) = B_t^{n,i}(B_{t+\varepsilon}^{n,j} - B_t^{n,j}),$$

and we conclude using the fact that Itô's integral is a limit in L^2 of Riemann sums. It follows that, almost surely,

$$\lim_{n\to\infty} \int_0^t B_s^{n,i} dB_s^{n,j} - B_s^{n,j} dB_s^{n,i} = \int_0^t B_s^i dB_s^j - B_s^j dB_s^i,$$

and we conclude that almost surely, $d(\mathbf{B}_t^n, \mathbf{B}_t) \to 0$. $\square$

The second lemma is a uniform Hölder estimate for $\mathbf{B}^n$.

Lemma 1.3. *For every $\alpha \in [0, 1/2)$, there exists a finite random variable K that belongs to L^p for every $p \ge 1$ and such that for every $0 \le s \le t \le T$, and every $n \ge 1$,*

$$d(\mathbf{B}_s^n, \mathbf{B}_t^n) \le K|t-s|^\alpha.$$

PROOF. By using the theorem of equivalence of norms, we see that there is a constant C such that

$$d(\mathbf{B}_s^n, \mathbf{B}_t^n) \le C\left(\|B_t^n - B_s^n\| + \sum_{i<j}\left|\int_s^t (B_u^{n,i} - B_s^{n,i}) dB_u^{n,j} - (B_u^{n,j} - B_s^{n,j}) dB_u^{n,i}\right|^{1/2}\right).$$

From the Garsia-Rodemich-Rumsey inequality, we know that there is a finite random variable K_1 (that belongs to L^p for every $p \ge 1$), such that for every $0 \le s \le t \le T$,

$$\left|\int_s^t (B_u^i - B_s^i) dB_u^j - (B_u^j - B_s^j) dB_u^i\right| \le K_1 |t-s|^{2\alpha}.$$

Since

$$\mathbb{E}\left(\int_s^t (B_u^i - B_s^i) dB_u^j - (B_u^j - B_s^j) dB_u^i \mid \mathcal{F}_n\right) = \int_s^t (B_u^{n,i} - B_s^{n,i}) dB_u^{n,j} - (B_u^{n,j} - B_s^{n,j}) dB_u^{n,i},$$

we deduce that

$$\left| \int_s^t (B_u^{n,i} - B_s^{n,i}) dB_u^{n,j} - (B_u^{n,j} - B_s^{n,j}) dB_u^{n,i} \right| \le K_2 |t-s|^{2\alpha},$$

where K_2 is a finite random variable that belongs to L^p for every $p \ge 1$. Similarly, of course, we have

$$\| B_t^n - B_s^n \| \le K_3 |t-s|^{\alpha},$$

and this completes the proof. □

We are now in position to finish the proof that, almost surely, $d_{p-var,[0,T]}(\mathbf{B}^n, \mathbf{B}) \to 0$ if $2 < p < 3$. Indeed, if t_i is a subdivision of $[0,T]$, we have for $2 < p' < p$,

$$\sum_{k=0}^{n-1} d\left((\mathbf{B}_{t_i}^n)^{-1} \mathbf{B}_{t_{i+1}}^n, (\mathbf{B}_{t_i})^{-1} \mathbf{B}_{t_{i+1}} \right)^p \le d_{p'-var,[0,T]}(\mathbf{B}^n, \mathbf{B}) \left(\sup_{s,t} d\left((\mathbf{B}_s^n)^{-1} \mathbf{B}_t^n, (\mathbf{B}_s)^{-1} \mathbf{B}_t \right) \right)^{p-p'}$$

By using the second lemma, it is seen that $d_{p'-var,[0,T]}(\mathbf{B}^n, \mathbf{B})$ is bounded when $n \to \infty$ and by combining the first two lemmas we easily see that $\sup_{s,t} d\left((\mathbf{B}_s^n)^{-1} \mathbf{B}_t^n, (\mathbf{B}_s)^{-1} \mathbf{B}_t \right) \to 0$.

2. Signature of the Brownian motion

Since a d-dimensional Brownian motion $(B_t)_{t \ge 0}$ is a p-rough path for $p > 2$, we know how to give a sense to the signature of the Brownian motion. In particular, the iterated integrals at any order of the Brownian motion are well defined. It turns out that these iterated integrals do not coincide with iterated Itô's integrals but with iterated Stratonovitch integrals. We start with some reminders about Stratonovitch integration. Let $(B_t)_{t \ge 0}$ be a one dimensional Brownian motion defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$. Let $(\Theta_t)_{0 \le t \le T}$ be a $\mathcal{F}$ adapted process such that $\mathbb{E}\left(\int_0^T \Theta_s^2 ds \right) < +\infty$. The Stratonovitch integral of Θ against B can be defined as the limit in probability of the sums

$$\sum_{k=0}^{n-1} \frac{\Theta_{t_{k+1}^n} + \Theta_{t_k^n}}{2} (B_{t_{k+1}^n} - B_{t_k^n}),$$

where $0 = t_0^n \le t_1^n \le \cdots \le t_n^n = T$ is a sequence of subdivisions whose mesh goes to 0. This limit is denoted $\int_0^T \Theta_s \circ dB_s$ and does not depend on the choice of the subdivision. It is an easy exercise to see that the relation between Itô's integral and Stratonovitch's is given by:

$$\int_0^T \Theta_s \circ dB_s = \int_0^T \Theta_s dB_s + \frac{1}{2} \langle \Theta, B \rangle_T,$$

where $\langle \Theta, B \rangle_T$ is the quadratic covariation between Θ and B. If $(B_t)_{t \ge 0}$ is d dimensional Brownian motion, we can then inductively define the iterated Stratonovitch integrals $\int_{0 \le t_1 \le \ldots \le t_k \le t} \circ dB_{t_1}^{i_1} \cdots \circ dB_{t_k}^{i_k}$. The next theorem proves that the signature of the Brownian rough path is given by multiple Stratonovitch integrals.

Theorem 2.1. *If $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion, the signature of B as a rough path is the formal series:*

$$\begin{aligned}\mathfrak{S}(B)_t &= 1 + \sum_{k=1}^{+\infty} \int_{\Delta^k[0,t]} \circ dB^{\otimes k} \\ &= 1 + \sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \left(\int_{0\leq t_1 \leq \dots \leq t_k \leq t} \circ dB^{i_1}_{t_1} \cdots \circ dB^{i_k}_{t_k} \right) X_{i_1}\cdots X_{i_k}.\end{aligned}$$

Proof. Let us work on a fixed interval $[0,T]$ and consider a sequence D_n of subdivisions of $[0,T]$ such that $D_{n+1}\subset D_n$ and whose mesh goes to 0 when $n\to+\infty$. As in the previous lecture, we denote by B^n the piecewise linear process which is obtained from B by interpolation along the subdivision D_n, that is for $t_i^n \leq t \leq t_{i+1}^n$,

$$B_t^n = \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} B_{t_i} + \frac{t - t_i^n}{t_{i+1}^n - t_i^n} B_{t_{i+1}}.$$

We know from the previous lecture that B^n converges to B in the p-rough paths topology $2<p<3$. In particular all the iterated integrals $\int_{\Delta^k[s,t]} dB^{n,\otimes k}$ converge. We claim that actually,

$$\lim_{n\to\infty} \int_{\Delta^k[s,t]} dB^{n,\otimes k} = \int_{\Delta^k[0,t]} \circ dB^{\otimes k}.$$

Let us denote

$$\int_{\Delta^k[s,t]} \partial B^{\otimes k} = \lim_{n\to\infty} \int_{\Delta^k[s,t]} dB^{n,\otimes k}.$$

We are going to prove by induction on k that $\int_{\Delta^k[s,t]} \partial B^{\otimes k} = \int_{\Delta^k[s,t]} \circ dB^{\otimes k}$. We have

$$\begin{aligned}\int_0^T B_s^n \otimes dB_s^n &= \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} B_s^n \otimes dB_s^n \\ &= \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \left(\frac{t_{i+1}^n - s}{t_{i+1}^n - t_i^n} B_{t_i^n} + \frac{s - t_i^n}{t_{i+1}^n - t_i^n} B_{t_{i+1}^n} \right) ds \otimes \frac{B_{t_{i+1}^n} - B_{t_i^n}}{t_{i+1}^n - t_i^n} \\ &= \frac{1}{2} \sum_{i=0}^{n-1} \left(B_{t_{i+1}^n} - B_{t_i^n} \right) \otimes \left(B_{t_{i+1}^n} + B_{t_i^n} \right)\end{aligned}$$

By taking the limit when $t\to\infty$, we deduce therefore that $\int_{\Delta^2[0,T]} \partial B^{\otimes 2} = \int_{\Delta^2[0,T]} \circ dB^{\otimes 2}$. In the same way, we have for $0\leq s<t\leq T$, $\int_{\Delta^2[s,t]} \partial B^{\otimes 2} = \int_{\Delta^2[s,t]} \circ dB^{\otimes 2}$. Assume now by induction, that for every $0\leq s\leq t\leq T$ and $1\leq j\leq k$, $\int_{\Delta^k[s,t]} \partial B^{\otimes k} = \int_{\Delta^k[s,t]} \circ dB^{\otimes k}$. Let us denote

$$\Gamma_{s,t} = \int_{\Delta^{k+1}[s,t]} \partial B^{\otimes (k+1)} - \int_{\Delta^{k+1}[s,t]} \circ dB^{\otimes (k+1)}.$$

From the Chen's relations, we immediately see that

$$\Gamma_{s,u} = \Gamma_{s,t} + \Gamma_{t,u}.$$

Moreover, it is easy to estimate

$$\|\Gamma_{s,t}\| \le C\omega(s,t)^{\frac{k+1}{p}},$$

where $2 < p < 3$ and $\omega(s,t) = \|\mathbf{B}\|_{p-var,[s,t]}$, $\mathbf{B}$ being the lift of B in the free Carnot group of step 2. Indeed, the bound

$$\int_{\Delta^{k+1}[s,t]} \partial B^{\otimes(k+1)} \le C_1\omega(s,t)^{\frac{k+1}{p}},$$

comes from the continuity of Lyons' lift and the bound

$$\int_{\Delta^{k+1}[s,t]} \circ dB^{\otimes(k+1)} \le C_2\omega(s,t)^{\frac{k+1}{p}},$$

easily comes from the Garsia-Rodemich-Rumsey inequality. As a conclusion, we deduce that $\Gamma_{s,t} = 0$ which proves the induction. □

We finish this lecture by a very interesting probabilistic object, the expectation of the Brownian signature. If

$$Y = y_0 + \sum_{k=1}^{+\infty} \sum_{I\in\{1,...,d\}^k} a_{i_1,...,i_k} X_{i_1}...X_{i_k}.$$

is a random series, that is if the coefficients are real random variables defined on a probability space, we will denote

$$\mathbb{E}(Y) = \mathbb{E}(y_0) + \sum_{k=1}^{+\infty} \sum_{I\in\{1,...,d\}^k} \mathbb{E}(a_{i_1,...,i_k}) X_{i_1}...X_{i_k}.$$

as soon as the coefficients of Y are integrable, where $\mathbb{E}$ stands for the expectation.

Theorem 2.2. *For $t \ge 0$,*

$$\mathbb{E}\left(\mathfrak{S}(B)_t\right) = \exp\left(t\left(\frac{1}{2}\sum_{i=1}^{d} X_i^2\right)\right).$$

Proof. An easy computation shows that if $\mathcal{I}_n$ is the set of words with length n obtained by all the possible concatenations of the words

$$\{(i,i)\}, \quad i \in \{1,...,d\},$$

(1) If $I \notin \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ dB^I\right) = 0;$$

(2) If $I \in \mathcal{I}_n$ then

$$\mathbb{E}\left(\int_{\Delta^n[0,t]} \circ dB^I\right) = \frac{t^{\frac{n}{2}}}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!},$$

Therefore,

$$\begin{aligned}\mathbb{E}\left(\mathfrak{S}(B)_t\right) &= 1 + \sum_{k=1}^{+\infty} \sum_{I \in \mathcal{I}_k} \frac{t^{\frac{k}{2}}}{2^{\frac{k}{2}} \left(\frac{k}{2}\right)!} X_{i_1} \dots X_{i_k} \\ &= \exp\left(t\left(\frac{1}{2}\sum_{i=1}^{d} X_i^2\right)\right).\end{aligned}$$

□

3. Stochastic differential equations as rough differential equations

Based on the results of the previous Lecture, it should come as no surprise that differential equations driven by the Brownian rough path should correspond to Stratonovitch differential equations. In this Lecture, we prove that it is indeed the case. Let us first remind to the reader the following basic result about existence and uniqueness for solutions of stochastic differential equations.

Let

$$(B_t)_{t \geq 0} = (B_t^1, ..., B_t^d)_{t \geq 0}$$

be a d-dimensional Brownian motion defined on some filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$.

Theorem 3.1. *Assume that $V_1, \cdots, V_d$ are C^2 vector fields with bounded derivatives up to order 2. Let $x_0 \in \mathbb{R}^n$. On $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, there exists a unique continuous and adapted process $(X_t)_{t \geq 0}$ such that for $t \geq 0$,*

$$X_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dB_s^i. \tag{3.2}$$

Thanks to Itô's formula the corresponding Itô's formulation is

$$X_t = x_0 + \frac{1}{2} \sum_{i=1}^{d} \int_0^t \nabla_{V_i} V_i(X_s) ds + \sum_{i=1}^{d} \int_0^t V_i(X_s) dB_s^i,$$

where for $1 \leq i \leq d$, $\nabla_{V_i} V_i$ is the vector field given by

$$\nabla_{V_i} V_i(x) = V_i^2 \mathbf{I}(x) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} v_i^k(x) \frac{\partial v_i^j}{\partial x_k}(x) \right) \frac{\partial}{\partial x_j}, \ x \in \mathbb{R}^n.$$

The main result of the Lecture is the following:

Theorem 3.2. *Let $\gamma > 2$ and let $V_1, \cdots, V_d$ be γ-Lipschitz vector fields on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$. The solution of the rough differential equation*

$$X_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s) \ dB_s^i,$$

is the solution of the Stratonovitch differential equation:

$$X_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dB^i_s.$$

PROOF. Let us work on a fixed interval $[0,T]$ and consider a sequence D_n of subdivisions of $[0,T]$ such that $D_{n+1} \subset D_n$ and whose mesh goes to 0 when $n \to +\infty$. As in the previous lectures, we denote by B^n the piecewise linear process which is obtained from B by interpolation along the subdivision D_n, that is for $t_i^n \le t \le t_{i+1}^n$,

$$B^n_t = \frac{t^n_{i+1} - t}{t^n_{i+1} - t^n_i} B_{t^n_i} + \frac{t - t^n_i}{t^n_{i+1} - t^n_i} B_{t^n_{i+1}}.$$

Let us then consider the process X_n that solves the equation

$$X^n_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X^n_s) \; dB^{i,n}_s,$$

and the process $\tilde{X}^n$, which is piecewise linear and such that

$$\tilde{X}^n_{t^n_{k+1}} = \tilde{X}^n_{t^n_k} + \sum_{i=i}^{d} V_i(X^n_{t^n_k})(B^i_{t^n_{k+1}} - B^i_{t^n_k}) + \frac{1}{2} \sum_{i=1}^{d} V_i^2 \mathbf{I}(X^n_{t^n_k})(t^n_{k+1} - t^n_k).$$

We can write

$$X_{t^n_{k+1}} - \tilde{X}_{t^n_{k+1}} = \sum_{\nu=0}^{k} (X_{t^n_{\nu+1}} - X_{t^n_\nu}) - (\tilde{X}_{t^n_{\nu+1}} - \tilde{X}_{t^n_\nu}).$$

Now,

$$\begin{aligned}(X_{t^n_{\nu+1}} - X_{t^n_\nu}) - (\tilde{X}_{t^n_{\nu+1}} - \tilde{X}_{t^n_\nu}) =& (X_{t^n_{\nu+1}} - X_{t^n_\nu}) - \sum_{i=i}^{d} V_i(X^n_{t^n_\nu})(B^i_{t^n_{\nu+1}} - B^i_{t^n_\nu}) \\ &- \frac{1}{2} \sum_{i=1}^{d} V_i^2 \mathbf{I}(X^n_{t^n_\nu})(t^n_{\nu+1} - t^n_\nu).\end{aligned}$$

From Davie's estimate, we have, with $2 < p < \gamma$,

$$\begin{aligned}&\left\| (X_{t^n_{\nu+1}} - X_{t^n_\nu}) - \sum_{i=i}^{d} V_i(X^n_{t^n_\nu})(B^i_{t^n_{\nu+1}} - B^i_{t^n_\nu}) - \sum_{i,j=1}^{d} (V_i V_j \mathbf{I})(X^n_{t^n_\nu}) \int_{t^n_\nu}^{t^n_{\nu+1}} (B^{n,i}_u - B^{n,i}_{t^n_\nu}) dB^{n,j}_u \right\| \\ \le& C \|V\|_{Lip^{\gamma-1}} \|S_2(B^n)\|^{\gamma}_{p-var,[t^n_\nu, t^n_{\nu+1}]} \\ \le& C \|V\|_{Lip^{\gamma-1}} \|B^n\|^{\gamma}_{p-var,[t^n_\nu, t^n_{\nu+1}]} \\ \le& C' \|V\|_{Lip^{\gamma-1}} \|B\|^{\gamma}_{p-var,[t^n_\nu, t^n_{\nu+1}]}.\end{aligned}$$

We deduce that, almost surely when $n \to \infty$,

$$\sum_{\nu=0}^{k} \left\| (X_{t^n_{\nu+1}} - X_{t^n_\nu}) - \sum_{i=i}^{d} V_i(X^n_{t^n_\nu})(B^i_{t^n_{\nu+1}} - B^i_{t^n_\nu}) - \sum_{i,j=1}^{d} (V_i V_j \mathbf{I})(X^n_{t^n_\nu}) \int_{t^n_\nu}^{t_{\nu+1}} (B^{n,i}_u - B^{n,i}_{t^n_\nu}) dB^{n,j}_u \right\| \to 0.$$

On the other hand,

$$\begin{aligned}
&\left\| \sum_{i,j=1}^{d} (V_i V_j \mathbf{I})(X^n_{t^n_\nu}) \int_{t^n_\nu}^{t^n_{\nu+1}} (B^{n,i}_u - B^{n,i}_{t^n_\nu}) dB^{n,j}_u - \frac{1}{2} \sum_{i=1}^{d} V_i^2 \mathbf{I}(X^n_{t^n_\nu})(t^n_{\nu+1} - t^n_\nu) \right\| \\
\le & \|V\|_{Lip^\gamma} \sum_{i,j=1}^{d} \left| \int_{t^n_\nu}^{t^n_{\nu+1}} (B^{n,i}_u - B^{n,i}_{t^n_\nu}) dB^{n,j}_u - \frac{1}{2}\delta_{ij}(t^n_{\nu+1} - t^n_\nu) \right| \\
\le & \frac{1}{2} \|V\|_{Lip^\gamma} \sum_{i,j=1}^{d} \left| (B^{n,i}_{t^n_{\nu+1}} - B^{n,i}_{t^n_\nu})(B^{n,j}_{t^n_{\nu+1}} - B^{n,j}_{t^n_\nu}) - \delta_{ij}(t^n_{\nu+1} - t^n_\nu) \right|
\end{aligned}$$

We deduce that in probability,

$$\sum_{\nu=0}^{k} \left\| \sum_{i,j=1}^{d} (V_i V_j \mathbf{I})(X^n_{t^n_\nu}) \int_{t^n_\nu}^{t^n_{\nu+1}} (B^{n,i}_u - B^{n,i}_{t^n_\nu}) dB^{n,j}_u - \frac{1}{2} \sum_{i=1}^{d} V_i^2 \mathbf{I}(X^n_{t^n_\nu})(t^n_{\nu+1} - t^n_\nu) \right\| \to 0.$$

We conclude that in probability,

$$X_{t^n_{k+1}} - \tilde{X}_{t^n_{k+1}} \to 0.$$

Up to an extraction of subsequence, we can assume that almost surely

$$X_{t^n_{k+1}} - \tilde{X}_{t^n_{k+1}} \to 0.$$

We now know that from the Lyons' continuity theorem, almost surely $X^n_t \to X_t$ where $(X_t)_{t\in[0,T]}$ is the solution of the rough differential equation

$$X_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s)\ dB^i_s.$$

Thus almost surely, we have that $\tilde{X}^n_t \to X_t$. On the othe hand, by definition, we have

$$\tilde{X}^n_{t^n_{k+1}} = \tilde{X}^n_{t^n_k} + \sum_{i=i}^{d} V_i(X^n_{t^n_k})(B^i_{t^n_{k+1}} - B^i_{t^n_k}) + \frac{1}{2} \sum_{i=1}^{d} V_i^2 \mathbf{I}(X^n_{t^n_k})(t^n_{k+1} - t^n_k).$$

which easily implies that $\tilde{X}^n$ converges in probability to $x_0 + \sum_{i=i}^d \int_0^t V_i(X_s) \circ dB^i_s$. This proves that

$$X_t = x_0 + \sum_{i=1}^{d} \int_0^t V_i(X_s) \circ dB^i_s.$$

□

4. The Stroock-Varadhan support theorem

To conclude this course, we are going to provide an elementary proof of the Stroock-Varadhan support theorem. We first remind that the support of a random variable X which defined on a metric space X is the smallest closed F such that $\mathbb{P}(X \in F) = 1$. In particular $x \in F$ if and only if for every open ball $\mathbf{B}(x,\varepsilon)$, $\mathbb{P}(X \in \mathbf{B}(x,\varepsilon)) > 0$.

Let $(B_t)_{0 \le t \le T}$ be a d-dimensional Brownian motion. We can see B as a random variable that takes its values in $C^{p-var}([0,T],\mathbb{R}^d)$, $p>2$. The following theorem describes the support of this random variable.

Proposition 4.1. *Let $p>2$. The support of B in $C^{p-var}([0,T],\mathbb{R}^d)$ is $C_0^{0,p-var}([0,T],\mathbb{R}^d)$, that is the closure for the p-variation distance of the set of smooth paths starting at 0.*

Proof. The key argument is a clever application of the Cameron-Martin theorem. Let us recall that this theorem says that if

$$h \in \mathbb{W}_0^{1,2} = \left\{ h : [0,T] \to \mathbb{R}^d, \exists k \in L^2([0,T],\mathbb{R}^d), h(t) = \int_0^t k(s)ds \right\},$$

then the distribution of $B+h$ is equivalent to the distribution of B.

Let us denote by F the support of B. It is clear that $F \subset C_0^{0,p-var}([0,T],\mathbb{R}^d)$, because the paths of B have bounded q variation for $2<q<p$. Let now $x \in F$. We have for $\varepsilon > 0$, $\mathbb{P}(d_{p-var}(B,x) < \varepsilon) > 0$. From the Cameron Martin theorem, we deduce then for $h \in \mathbb{W}_0^{1,2}$, $\mathbb{P}(d_{p-var}(B+h,x) < \varepsilon) > 0$. This shows that $x-h \in F$. We can find a sequence of smooth x_n that converges to x in p-variation. From the previous argument $x - x_n \in F$ and converges to 0. Thus $0 \in F$ and using the same argument shows then that $\mathbb{W}_0^{1,2}$ is included in F. This proves that $F = C_0^{0,p-var}([0,T],\mathbb{R}^d)$. □

The following theorem due to Stroock and Varadhan describes the support of solutions of stochastic differential equations. As in the previous proof, we denote

$$\mathbb{W}_0^{1,2} = \left\{ h : [0,T] \to \mathbb{R}^d, \exists k \in L^2([0,T],\mathbb{R}^d), h(t) = \int_0^t k(s)ds \right\}.$$

Theorem 4.2. *Let $\gamma > 2$ and let $V_1, \cdots, V_d$ be γ-Lipschitz vector fields on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$. Let $(X_t)_{t \ge 0}$ be the solution of the Stratonovitch differential equation:*

$$X_t = x_0 + \sum_{i=1}^d \int_0^t V_i(X_s) \circ dB_s^i.$$

Let $p>2$. The support of X in $C^{p-var}([0,T],\mathbb{R}^d)$ is the closure in the p-variation topology of the set:

$$\left\{ x^h, h \in \mathbb{W}_0^{1,2} \right\},$$

where x^h is the solution of the ordinary differential equation

$$x_t^h = x_0 + \sum_{i=1}^d \int_0^t V_i(x_s^h) dh_s^i$$

Proof. We denote by B^n the piecewise linear process which is obtained from B by interpolation along a subdivision D_n which is such that $D_{n+1} \subset D_n$ and whose mesh goes to 0. We know that $B^n \in \mathbb{W}_0^{1,2}$ and that x^{B^n} almost surely converges in p-variation to X. As a consequence B almost surely takes its values in the closure of:

$$\left\{ x^h, h \in \mathbb{W}_0^{1,2} \right\}.$$

This shows that the support of B is included in the closure of $\left\{x^h, h \in \mathbb{W}_0^{1,2}\right\}$. The converse inclusion is a little more difficult and relies on the Lyons' continuity theorem. It can be proved by using similar arguments as for B (details are let to the reader) that the support of $S_2(B)$ is the is the closure in the p-variation topology of the set:

$$\left\{S_2(h), h \in \mathbb{W}_0^{1,2}\right\},$$

where S_2 denotes, as usual, the lift in the Carnot group of step 2. Take $h \in \mathbb{W}_0^{1,2}$ and $\varepsilon > 0$. By the Lyons' continuity theorem, there exists therefore $\eta > 0$ such that $d_{p-var}(S_2(h), S_2(B)) < \eta$ implies $\|X - x^h\|_{p-var} < \varepsilon$. Therefore

$$0 < \mathbb{P}\left(d_{p-var}(S_2(h), S_2(B)) < \eta\right) \leq \mathbb{P}\left(\|X - x^h\|_{p-var} < \varepsilon\right).$$

In particular, we have $\mathbb{P}\left(\|X - x^h\|_{p-var} < \varepsilon\right) > 0$. This proves that x^h is in the support of X. So, the proof now boils down to the statement that the support of $S_2(B)$ is the closure in the p-variation topology of the set:

$$\left\{S_2(h), h \in \mathbb{W}_0^{1,2}\right\}.$$

□