

MATHEMATICS AS A CREATIVE ART

by P. R. Halmos

What is mathematics? Different people would give different answers. A student in elementary school would probably say that it was about adding, subtracting, multiplying and dividing. Oh, yes—and about fractions and decimals too. A student in high school would probably say that it is about learning rules and formulas to solve equations. Oh, yes—and learning rules and formulas in geometry too. I'm afraid that all too many students of calculus would also say that mathematics is about rules and formulas and impossible word problems and getting the right answer by the right method. Then, since most people lose contact with mathematics after high school, or after calculus, the average citizen keeps a limited view of mathematics for a lifetime. That is too bad, because those answers are not complete and we should not carry around in our heads any more delusions or distorted views of reality than we have to.

The following selection gives an answer by a professional mathematician. It is not the answer that would be given by every professional mathematician, but it is probably far closer to the truth than the answers that people in general would give. It is well worth the attention of every student of calculus, especially those who will not become professional mathematicians. Most students of calculus will forget how to find equations of tangent lines, but they should remember that mathematics is far closer to an art than it is to the business of equation-solving.

Do you know any mathematicians—and, if you do, do you know anything about what they do with their time? Most people don't. When I get into conversation with the man next to me in a plane, and he tells me that he is something respectable like a doctor, lawyer, merchant, or dean, I am tempted to say that I am in roofing and siding. If I tell him that I am a mathematician, his most likely reply will be that he himself could never balance his check book, and it must be fun to be a whiz at math. If my neighbor is an astronomer, a biologist, a chemist, or any other kind of natural or social scientist, I am, if anything, worse off—this man *thinks* he knows what a mathematician is, and he is probably wrong. He thinks that I spend my time (or should) converting different orders of magnitude, comparing binomial coefficients and powers of 2, or solving equations involving rates of reactions.

C. P. Snow points to and deplores the existence of two cultures; he worries about the physicist whose idea of modern literature is Dickens, and he chides the poet who cannot state the second law of thermodynamics. Mathematicians, in converse with well-meaning, intelligent, and educated laymen (do you mind if I refer to all non-mathematicians as laymen?) are much worse off than physicists in converse with poets. It saddens me that educated people don't even know that my subject exists. There is something that they call mathematics, but they neither know how the professionals use that word, nor can they conceive why anybody should do it.

It is, to be sure, possible that an intelligent and otherwise educated person doesn't know that egyptology exists, or hematology, but all you have to tell him is that it does, and he will immediately understand in a rough general way why it should and he will have some empathy with the scholar of the subject who finds it interesting.

Usually when a mathematician lectures, he is a missionary. Whether he is talking over a cup of coffee with a collaborator, lecturing to a graduate class of specialists, teaching a reluctant group of freshman engineers, or addressing a general audience of laymen—he is still preaching and seeking to make converts. He will state theorems and he will discuss proofs and he will hope that when he is done his audience will know more mathematics than they did before. My aim today is different—I am not here to proselyte but to enlighten—I seek not converts but friends. I do not want to teach you what mathematics is, but only *that* it is.

I call my subject mathematics—that's what all my colleagues call it, all over the world—and there, quite possibly, is the beginning of confusion. The word covers two disciplines—many more, in reality, but two, at least two, in the same sense in which Snow speaks of two cultures. In order to have some words with which to refer to the ideas I want to discuss, I offer two temporary and ad hoc neologisms. Mathematics, as the word is customarily used, consists of at least two



distinct subjects, and I propose to call them *mathology* and *mathophysics*. Roughly speaking, mathology is what is usually called pure mathematics, and mathophysics is called applied mathematics, but the qualifiers are not emotionally strong enough to disguise that they qualify the same noun. If the concatenation of syllables I chose here reminds you of other words, no great harm will be done; the rhymes alluded to are not completely accidental. I originally planned to entitle this lecture something like "Mathematics is an art," or "Mathematics is not a science," or "Mathematics is useless," but the more I thought about it the more I realized that I mean that "Mathology is an art," "Mathology is not a science," and "Mathology is useless." When I am through, I hope you will recognize that most of you have known about mathophysics before, only you were probably calling it mathematics; I hope that all of you will recognize the distinction between mathology and mathophysics; and I hope that some of you will be ready to embrace, or at least applaud, or at the very least, recognize mathology as a respectable human endeavor.

In the course of the lecture I'll have to use many analogies (literature, chess, painting), each imperfect by itself, but I hope that in their totality they will serve to delineate what I want delineated. Sometimes in the interest of economy of time, and sometimes doubtless unintentionally, I'll exaggerate; when I'm done, I'll be glad to rescind anything that was inaccurate or that gave offense in any other way.

What Mathematicians Do

As a first step toward telling you what mathematicians do, let me tell you some of the things they do not do. To begin with, mathematicians have very little to do with numbers. You can no more expect a mathematician to be able to add a column of figures rapidly and correctly than you can expect a painter to draw a straight line or a surgeon to carve a turkey—popular legend attributes such skills to these professions, but popular legend is wrong. There is, to be sure, a part of mathematics called number theory, but even that doesn't deal with numbers in the legendary sense—a number theorist and an adding machine would find very little to talk about. A machine might enjoy proving that $1^3 + 5^3 + 3^3 = 153$ and it might even go on to discover that there are only five positive integers with the property that the equation indicates (1, 370, 371, 407), but most mathematicians couldn't care less; many mathematicians enjoy and respect the theorem that every positive integer is the sum of not more than four squares, whereas the infinity involved in the word "every" would frighten and

paralyze any ordinary office machine, and, in any case, that's probably not the sort of thing that the person who relegates mathematicians to numbers had in mind.

Not even those romantic objects of latter day science fiction, the giant brains, the computing machines that run our lives these days—not even they are of interest to the mathematician as such. Some mathematicians are interested in the logical problems involved in the reduction of difficult questions to the sort of moronic baby talk that machines understand: the logical design of computing machines is definitely mathematics. Their construction is not, that's engineering, and their product, be it a payroll, a batch of sorted mail, or a supersonic plane, is of no mathematical interest or value.

Mathematics is not numbers or machines; it is also not the determination of the heights of mountains by trigonometry, or compound interest by algebra, or moments of inertia by calculus. Not today it isn't. At one point in history each of those things, and others like them, might have been an important and non-trivial research problem, but once the problem is solved, its repetitive application has as much to do with mathematics as the work of a Western Union messenger boy has to do with Marconi's genius.

There are at least two other things that mathematics isn't; one of them is something it never was, and the other is something it once included and by now has sloughed off. The first is physics. Some laymen confuse mathematics and theoretical physics and speak, for instance, of Einstein as a great mathematician. There is no doubt that Einstein was a great man, but he was no more a great mathematician than he was a great violinist. He used mathematics to find out facts about the universe, and that he successfully used certain parts of differential geometry for that purpose adds a certain piquancy to the appeal of differential geometry. Withal, relativity theory and differential geometry are not the same thing. Einstein, Schrodinger, Heisenberg, Fermi, Wigner, Feynman—great men all, but not mathematicians; some of them, in fact, strongly antimathematical, preach against mathematics, and would regard it as an insult to be called a mathematician.

What once was mathematics remains mathematics always, but it can become so thoroughly worked out, so completely understood, and, in the light of millennia of contributions, with hindsight, so trivial, that mathematicians never again need to or want to spend time on it. The celebrated Greek problems (trisect the angle, square the circle, duplicate the cube) are of this kind, and the irrepressible mathematical amateur to the contrary notwithstanding, mathematicians are no longer trying to solve them. Please understand, it isn't that they have given up. Perhaps you have heard that,

according to mathematicians, it is impossible to square a circle, or trisect an angle, and perhaps you have heard or read that, therefore, mathematicians are a pusillanimous chicken-hearted lot, who give up easily, and use their ex-cathedra pronouncements to justify their ignorance. The conclusion may be true, and you may believe it if you like, but the proof is inadequate.

The Start of Mathematics

No one knows when and where mathematics got started, or how, but it seems reasonable to guess that it emerged from the same primitive physical observations (counting, measuring) with which we all begin our own mathematical insight (ontogeny recapitulates phylogeny). It was probably so in the beginning, and it is true still, that many mathematical ideas originate not from pure thought but from material necessity; many, but probably not all. Almost as soon as a human being finds it necessary to count his sheep (or sooner?) he begins to wonder about numbers and shapes and motions and arrangements—curiosity about such things seems to be as necessary to the human spirit as curiosity about earth, water, fire, and air, and curiosity—sheer pure intellectual curiosity—about stars and about life. Numbers and shapes and motions and arrangements, and also thoughts and their order, and concepts such as "property" and "relation"—all such things are the raw material of mathematics. The technical but basic mathematical concept of "group" is the best humanity can do to understand the intuitive concept of "symmetry" and the people who study topological spaces, and ergodic paths, and oriented graphs are making precise our crude and vague feelings about shapes, and motions, and arrangements.

Why do mathematicians study such things, and why should they? What, in other words, motivates the individual mathematician, and why does society encourage his efforts, at least to the extent of providing him with the training and subsequently the livelihood that, in turn, give him the time he needs to think? There are two answers to each of the two questions: because mathematics is practical and because mathematics is an art. The already existing mathematics has more and more new applications each day, and the rapid growth of desired applications suggests more and more new practical mathematics. At the same time, as the quantity of mathematics grows and the number of people who think about it keeps doubling over and over again, more new concepts need explication, more new logical interrelations cry out for study, and understanding, and simplification, and more and more the tree of mathematics bears elaborate and gaudy flowers that are, to many

beholders, worth more than the roots from which it all comes and the causes that brought it all into existence.

Mathematics is very much alive today. There are more than a thousand journals that publish mathematical articles; about 15,000 to 20,000 mathematical articles are printed every year. The mathematical achievements of the last 100 years are greater in quantity and in quality than those of all previous history. Difficult mathematical problems, which stumped Hilbert, Cantor, or Poincaré, are being solved, explained, and generalized by beardless (and bearded) youths in Berkeley and in Odessa.

Mathematicians sometimes classify themselves and each other as either problem-solvers or theory-creators. The problem-solvers answer yes-or-no questions and discuss the vital special cases and concrete examples that are the flesh and blood of mathematics; the theory-creators fit the results into a framework, illuminate it all, and point it in a definite direction—they provide the skeleton and the soul of mathematics. One and the same human being can be both a problem-solver and a theory-creator, but, usually, he is mainly one or the other. The problem-solvers make geometric constructions, the theory-creators discuss the foundations of Euclidean geometry; the problem-solvers find out what makes switching diagrams tick, the theory-creators prove representation theorems for Boolean algebras. In both kinds of mathematics and in all fields of mathematics the progress in one generation is breathtaking. No one can call himself a mathematician nowadays who doesn't have at least a vague idea of homological algebra, differential topology, and functional analysis, and every mathematician is probably somewhat of an expert on at least one of these subjects—and yet when I studied mathematics in the 1930's none of those phrases had been invented, and the subjects they describe existed in seminal forms only.

Mathematics is abstract thought, mathematics is pure logic, mathematics is creative art. All these statements are wrong, but they are all a little right, and they are all nearer the mark than "mathematics is numbers" or "mathematics is geometric shapes." For the professional pure mathematician, mathematics is the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof. Simplicity, intricacy, and above all, logical analysis are the hallmark of mathematics.

The mathematician is interested in extreme cases—in this respect he is like the industrial experimenter who breaks lightbulbs, tears shirts, and bounces cars on ruts. How widely does a reasoning apply, he wants to know, and what happens when it doesn't? What

happens when you weaken one of the assumptions, or under what conditions can you strengthen one of the conclusions? It is the perpetual asking of such questions that makes for broader understanding, better technique, and greater elasticity for future problems.

Mathematics—this may surprise you or shock you some—is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof. The conviction is not likely to come early—it usually comes after many attempts, many failures, many discouragements, many false starts. It often happens that months of work result in the proof that the method of attack they were based on cannot possibly work, and the process of guessing, visualizing, and conclusion-jumping begins again. A reformulation is needed and—and this too may surprise you—more experimental work is needed. To be sure, by "experimental work" I do not mean test tubes and cyclotrons. I mean thought-experiments. When a mathematician wants to prove a theorem about an infinite-dimensional Hilbert space, he examines its finite-dimensional analogue, he looks in detail at the 2- and 3-dimensional cases, he often tries out a particular numerical case, and he hopes that he will gain thereby an insight that pure definition-juggling has not yielded. The deductive stage, writing the result down, and writing down its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman's work, not the architect's.

Mathematics is a Language

Why does mathematics occupy such an isolated position in the intellectual firmament? Why is it good form, for intellectuals, to shudder and announce that they can't bear it, or, at the very least, to giggle and announce that they never could understand it? One reason, perhaps, is that mathematics is a language. Mathematics is a precise and subtle language designed to express certain kinds of ideas more briefly, more accurately, and more usefully than ordinary language. I do not mean here that mathematicians, like members of all other professional cliques, use jargon. They do, at times, and they don't most often, but that's a personal phenomenon, not the professional one I am describing. What I do mean by saying that mathematics is a language is sketchily and inadequately illustrated by the difference between the following two sentences. (1) If each of two numbers is multiplied by itself, the difference of the two results is the same as the product of the

sum of the two given numbers by their difference. (2) $x^2 - y^2 = (x + y)(x - y)$ (Note: the longer formulation is not only awkward, it is also incomplete.)

One thing that sometimes upsets and repels the layman is the terminology that mathematicians employ. Mathematical words are intended merely as labels, sometimes suggestive, possibly facetious, but always precisely defined; their everyday connotations must be steadfastly ignored. Just as nobody nowadays infers from the name Fitzgerald that its bearer is the illegitimate son of Gerald, a number that is called irrational must not be thought unreasonable; just as a dramatic poem called *The Divine Comedy* is not necessarily funny, a number called imaginary has the same kind of mathematical existence as any other. (Rational, for numbers, refers not to the Latin *ratio*, in the sense of reason, but to the English "ratio," in the sense of quotient.)

Mathematics is a language. None of us feels insulted when a sinologist uses Chinese phrases, and we are resigned to living without Chinese, or else spending years learning it. Our attitude to mathematics should be the same. It's a language, and it takes years to learn to speak it well. We all speak it a little, just because some of it is in the air all the time, but we speak it with an accent, and frequently inaccurately; most of us speak it, say, about as well as one who can only say "*Oui, monsieur*" and "*S'il vous plaît*" speaks French. The mathematician sees nothing wrong with this as long as he's not upbraided by the rest of the intellectual community for keeping secrets. It took him a long time to learn his language, and he doesn't look down on the friend who, never having studied it, doesn't speak it. It is however sometimes difficult to keep one's temper with the cocktail party acquaintance who demands that he be taught the language between drinks and who regards failure or refusal to do so as sure signs of stupidity or snobbishness.

Some Analogies

A little feeling for the nature of mathematics and mathematical thinking can be got by the comparison with chess. The analogy, like all analogies, is imperfect, but it is illuminating just the same. The rules for chess are as arbitrary as the axioms of mathematics sometimes seem to be. The game of chess is as abstract as mathematics. (That chess is played with solid pieces, made of wood, or plastic, or glass, is not an intrinsic feature of the game. It can just as well be played with pencil and paper, as mathematics is, or blindfolded, as mathematics can.) Chess also has its elaborate technical language, and chess is completely deterministic.

There is also some analogy between mathematics and music. The mathologist feels the need to justify pure mathematics exactly as little as the musician feels the need to justify music. Do practical men, the men who meet payrolls, demand only practical music—soothing jazz to make an assembly line worker turn nuts quicker, or stirring marches to make a soldier kill with more enthusiasm? No, surely none of us believes in that kind of justification; music, and mathematics, are of human value because human beings feel they are.

The analogy with music can be stretched a little further. Before a performer's artistic contribution is judged, it is taken for granted that he hits the right notes, but merely hitting the right notes doesn't make him a musician. We don't get the point of painting if we compliment the nude Maya on being a good likeness, and we don't get the point of a historian's work if all we can say is that he didn't tell lies. Mere accuracy in performance, resemblance in appearance, and truth in storytelling doesn't make good music, painting, history: in the same way, mere logical correctness doesn't make good mathematics.

Goodness, high quality, are judged on grounds more important than validity, but less describable. A good piece of mathematics is connected with much other mathematics, it is new without being silly (think of a "new" western movie in which the names and the costumes are changed, but the plot isn't), and it is deep in an ineffable but inescapable sense—the sense in which Johann Sebastian is deep and Carl Philip Emmanuel is not. The criterion for quality is beauty, intricacy, neatness, elegance, satisfaction, appropriateness—all subjective, but all somehow mysteriously shared by all.

Mathematics resembles literature also, differently from the way it resembles music. The writing and reading of literature are related to the writing and reading of newspapers, advertisements, and road signs the way mathematics is related to practical arithmetic. We all need to read and write and figure for daily life: but literature is more than reading and writing, and mathematics is more than figuring. The literature analogy can be used to help understand the role of teachers and the role of the pure-applied dualism.

Many whose interests are in language, in the structure, in the history, and in the aesthetics of it, earn their bread and butter by teaching the rudiments of language to its future practical users. Similarly many, perhaps most, whose interests are in the mathematics of today, earn their bread and butter by teaching arithmetic, trigonometry, or calculus. This is sound economics: society abstractly and impersonally is willing to subsidize pure language and pure mathematics, but not very far. Let the would-be purist pull his weight by teaching

the next generation the applied aspects of his craft; then he is permitted to spend a fraction of his time doing what he prefers. From the point of view of what a good teacher must be, this is good. A teacher must know more than the bare minimum he must teach; he must know more in order to avoid more and more mistakes, to avoid the perpetuation of misunderstanding, to avoid catastrophic educational inefficiency. To keep him alive, to keep him from drying up, his interest in syntax, his burrowing in etymology, or his dabbling in poetry play a necessary role.

The pure-applied dualism exists in literature too. The source of literature is human life, but literature is not the life it comes from, and writing with a grim purpose is not literature. Sure there are borderline cases: is Upton Sinclair's *Jungle* literature or propaganda? (For that matter, is Chiquita Banana an advertising jingle or charming light opera?) But the fuzzy boundary doesn't alter the fact that in literature (as in mathematics) the pure and the applied are different in intent, in method, and in criterion of success.

Perhaps the closest analogy is between mathematics and painting. The origin of painting is physical reality, and so is the origin of mathematics—but the painter is not a camera and the mathematician is not an engineer. The painter of "Uncle Sam Wants You" got his reward from patriotism, from increased enlistments, from winning the war—which is probably different from the reward Rembrandt got from a finished work. How close to reality painting (and mathematics) should be is a delicate matter of judgment. Asking a painter to "tell a concrete story" is like asking a mathematician to "solve a real problem." Modern painting and modern mathematics are far out—too far in the judgment of some. Perhaps the ideal is to have a spice of reality always present, but not to crowd it the way descriptive geometry, say, does in mathematics, and medical illustration, say, does in painting.

Talk to a painter (I did) and talk to a mathematician, and you'll be amazed at how similarly they react. Almost every aspect of the life and of the art of a mathematician has its counterpart in painting, and vice versa. Every time a mathematician hears "I could never make my checkbook balance" a painter hears "I could never draw a straight line"—and the comments are equally relevant and equally interesting. The invention of perspective gave the painter a useful technique, as did the invention of 0 to the mathematician. Old art is as good as new; old mathematics is as good as new. Tastes change, to be sure, in both subjects, but a twentieth century painter has sympathy for cave paintings and a twentieth century mathematician for the fraction juggling of the Babylonians. A painting must

be painted and then looked at; a theorem must be printed and then read. The painter who thinks good pictures, and the mathematician who dreams beautiful theorems are dilettantes; an unseen work of art is incomplete. In painting and in mathematics there are some objective standards of good—the painter speaks of structure, line, shape, and texture, where the mathematician speaks of truth, validity, novelty, generality—but they are relatively the easiest to satisfy. Both painters and mathematicians debate among themselves whether these objective standards should even be told to the young—the beginner may misunderstand and overemphasize them and at the same time lose sight of the more important subjective standards of goodness. Painting and mathematics have a history, a tradition, a growth. Students, in both subjects, tend to flock to the newest but, except the very best, miss the point; they lack the vitality of what they imitate, because, among other reasons, they lack the experience based on the traditions of the subject.

I've been talking *about* mathematics, but not *in* it, and, consequently, what I've been saying is not capable of proof in the mathematical sense of the word. I hope just the same, that I've shown you that there is a subject called mathematics (mathology?), and that that subject is a creative art. It is a creative art because mathematicians create beautiful new concepts; it is a creative art because mathematicians live, act, and think like artists; and it is a creative art because mathematicians regard it so. I feel strongly about that, and I am grateful for this opportunity to tell you about it. Thank you for listening.

carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof." Now there is a definition! It is certainly not what your ordinary person would respond if asked to define mathematics. How could you take the four elements of the definition—"logical dovetailing," "set of assumptions," "surprising conclusions," and "elegant proof"—and make them plain to an educated person?

5. Can the distinction made between *mathology* and *mathophysics* also be made in other areas of human endeavor? If so, what are some examples; if not, why not?

QUESTIONS AND EXERCISES

1. "It saddens me that educated people don't even know that my subject exists." No doubt, but so what? There are people who spend their days valuing the reserves of group annuities, and educated people don't know anything about *that* either. Should educated people care? Or does the author just have a big ego and wants everybody to know what *he* does?

2. $1^3 + 5^3 + 3^3 = 153$ all right. Why did the author give no examples with squares? Clearly, because there are none. *Prove* that this is so. That is, show that $a^2 + b^2 = 10a + b$ has no solutions if a and b are digits and a is not zero.

3. If you solved that problem, did you enjoy it? If so, how, and why? If not, why not? What sorts of pleasure can come from mathematics?

4. "Mathematics is the logical dovetailing of a