Geometric Inequalities on sub-Riemannian manifolds, Lecture Notes Tata Insitute 2018

Fabrice Baudoin

Department of Mathematics, University of Connecticut,
341 Mansfield Road, Storrs, CT 06269-1009, USA
`fabrice.baudoin@uconn.edu`

January 8, 2018

Abstract

We present recent developments in the geometric analysis of sub-Laplacians on sub-Riemannian manifolds.

Contents

1

1 Introduction

Study of sub-Riemannian manifolds. Focus on:

1. Sub-Laplacian comparison theorems

2. Volume doubling properties

3. Isoperimetric and Sobolev inequalities

Different sets of techniques:

1. Lagrangian methods: Jacobi fields

2. Eulerian methods: Bochner's identities

2 Sub-Laplacian comparison theorems on Sasakian manifolds

This section is mostly based on [17], to which we refer for details and statements which are not proved here.

2.1 Laplacian comparison theorems on Riemannian manifolds

In what follows, given $\kappa \in \mathbb{R}$, we will indicate with $\mathbb{M}_\kappa$ the simply connected Riemannian space form of constant sectional curvature κ. We recall that:

$$\mathbb{M}_\kappa = \begin{cases} \text{the sphere } \mathbb{S}^n \text{ of constant sectional curvature } \kappa, \text{ when } \kappa > 0; \\ \mathbb{R}^n, \text{ when } \kappa = 0; \\ \text{the hyperbolic space } \mathbb{H}^n \text{ of constant sectional curvature } \kappa, \text{ when } \kappa < 0. \end{cases}$$

The next result plays a key role in Riemannian geometry.

Theorem 2.1 (Laplacian comparison theorem). *Let $\mathbb{M}$ be a n-dimensional complete Riemannian manifold with*

$$\text{Ric}(\mathbb{M}) \geq (n-1)\kappa, \qquad (\kappa \in \mathbb{R})$$

and denote by $\mathbb{M}_\kappa$ the n-dimensional simply connected space of constant sectional curvature κ. Let $r_{\mathbb{M}}$ and $r_{\mathbb{M}_\kappa}$ respectively denote the geodesic distances from some fixed points in $\mathbb{M}$ and $\mathbb{M}_\kappa$. If $x \in \mathbb{M}$ is such that $r_{\mathbb{M}}$ is differentiable at x, then for any $y \in \mathbb{M}_\kappa$ at which $r_{\mathbb{M}}(x) = r_{\mathbb{M}_\kappa}(y)$, we have

$$\Delta_{\mathbb{M}} r_{\mathbb{M}}(x) \leq \Delta_{\mathbb{M}_\kappa} r_{\mathbb{N}}(y). \tag{2.1}$$

The inequality (2.1) *can be equivalently written*

$$\Delta_{\mathbb{M}} r_{\mathbb{M}}(x) \leq \begin{cases} (n-1)\sqrt{\kappa}\cot(\sqrt{\kappa}r_{\mathbb{M}}(x)), & \text{if } \kappa > 0, \\ \frac{n-1}{r_{\mathbb{M}}(x)}, & \text{if } \kappa = 0, \\ (n-1)\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}r_{\mathbb{M}}(x)), & \text{if } \kappa < 0. \end{cases} \tag{2.2}$$

On the whole manifold $\mathbb{M}$ the inequality (3.3) *holds in the sense of the distributions.*

Several proofs:

- Jacobi fields

- Bochner's formula

2.2   Sub-Riemannian manifolds

Let $\mathbb{M}$ be a smooth, connected manifold with dimension $n + m$. We assume that $\mathbb{M}$ is equipped with a bracket generating sub-bundle $\mathcal{H} \subset \mathbf{T}\mathbb{M}$ of rank n and a fiberwise inner product $g_{\mathcal{H}}$ on that distribution.

- The distribution $\mathcal{H}$ is referred to as the set of *horizontal directions*.

- Sub-Riemannian geometry is the study of the geometry which is intrinsically associated to $(\mathcal{H}, g_{\mathcal{H}})$.

In sub-Riemannian geometry, for sub-Laplacians comparison theorems the situation is much more difficult. Though, there is a sub-Riemannian distance

$$d(x, y) = \inf_{\gamma \in C(x,y)} \int_0^1 \sqrt{g_{\mathcal{H}}(\gamma'(t), \gamma'(t))} dt$$

and in many situations a natural sub-Laplacian, several questions arise:

1. What is the analogue of $\text{Ric}(\mathbb{M}) \geq (n-1)\kappa$?

2. What is a sub-Riemannian space form ?

Already in the lowest possible dimensional case 3, the situation is difficult and was first addressed by Agrachev-Lee [2]. Of course, one should think of the Heisenberg group as a space form with zero curvature, but the sub-Laplacian $\Delta_{\mathcal{H}}$ of the sub-Riemannian distance function r turns out to have an ugly form. Actually $\Delta_{\mathcal{H}} r$ is not a function of r only, but of r and Zr, and this function though explicit is not as simple as one could hope (Agrachev-Lee [2]):

$$\Delta_{\mathcal{H}} r = F_H(r, Zr)$$

with

$$F(r, v) = \begin{cases} \frac{rv(\sin(rv) - rv\cos(rv))}{r(2 - 2\cos(rv) - rv\sin(rv))}, & v > 0 \\ \frac{4}{r} \end{cases}$$

But, it turns out that $F_H(r, Zr) \leq F(r, 0) = \frac{4}{r}$, so one gets the clean upper bound

$$\Delta_{\mathcal{H}} r \leq \frac{4}{r}.$$

The 4 is a little puzzling since it implies a measure contraction property $\mathbf{MCP}(0, 5)$, whereas the Hausdorff dimension of the Heisenberg group is 4 !

Agrachev-Lee proves then that for 3-dimensional Sasakian manifolds with non-negative sectional curvature:

$$\Delta_{\mathcal{H}} r \leq F_H(r, Zr) \leq \frac{4}{r}.$$

The result was then extended to general Sasakian manifolds by Lee-Li.

2.3 Sasakian model spaces

We first present the sub-Riemannian Sasakian model spaces.

2.3.1 Heisenberg group

The Heisenberg group is the set

$$\mathbb{H}^{2n+1} = \{(x, y, z), x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}\}$$

endowed with the group law

$$(x_1, y_1, z_1) \star (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \langle x_1, y_2 \rangle_{\mathbb{R}^n} - \langle x_2, y_1 \rangle_{\mathbb{R}^n}).$$

The vector fields

$$X_i = \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial z}$$

$$Y_i = \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial z}$$

and
$$Z = \frac{\partial}{\partial z}$$

form an orthonormal frame of left invariant vector fields for the left invariant metric on $\mathbb{H}^{2n+1}$. Note that the following commutations hold

$$[X_i, Y_j] = 2\delta_{ij}Z, \quad [X_i, Z] = [Y_i, Z] = 0.$$

The map

$$\pi : \begin{array}{ccc} \mathbb{H}^{2n+1} & \rightarrow & \mathbb{R}^{2n} \\ (x, y, z) & \rightarrow & (x, y) \end{array}$$

is then a Riemannian submersion with totally geodesic fibers. The horizontal Laplacian is the left invariant operator

$$\Delta_{\mathcal{H}} = \sum_{i=1}^{n} (X_i^2 + Y_i^2)$$

$$= \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 2 \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) \frac{\partial}{\partial z} + (\|x\|^2 + \|y\|^2) \frac{\partial^2}{\partial z^2}$$

and the vertical Laplacian is the left invariant operator

$$\Delta_{\mathcal{V}} = \frac{\partial^2}{\partial z^2}.$$

The horizontal distribution

$$\mathcal{H} = \mathbf{span}\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$$

is bracket generating at every point, so $\Delta_{\mathcal{H}}$ is a subelliptic operator. The operator $\Delta_{\mathcal{H}}$ is invariant by the action of the orthogonal group of $\mathbb{R}^{2n}$ on the variables (x, y). Introducing the variable $r^2 = \|x\|^2 + \|y\|^2$, we see then that the radial part of $\Delta_{\mathcal{H}}$ is given by

$$\bar{\Delta}_{\mathcal{H}} = \frac{\partial^2}{\partial r^2} + \frac{2n - 1}{r} \frac{\partial}{\partial r} + r^2 \frac{\partial^2}{\partial z^2}.$$

This means that if $f : \mathbb{R}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ is a smooth map and ρ is the submersion $(x, y, z) \to (\sqrt{\|x\|^2 + \|y\|^2}, z)$ then

$$\Delta_{\mathcal{H}}(f \circ \rho) = (\bar{\Delta}_{\mathcal{H}} f) \circ \rho.$$

2.3.2 The Hopf fibration

Let us consider the odd dimensional unit sphere

$$\mathbb{S}^{2n+1} = \{z = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}, \|z\| = 1\}.$$

There is an isometric group action of $\mathbb{S}^1 = \mathbf{U}(1)$ on $\mathbb{S}^{2n+1}$ which is defined by

$$(z_1, \cdots, z_n) \rightarrow (e^{i\theta} z_1, \cdots, e^{i\theta} z_n).$$

The quotient space $\mathbb{S}^{2n+1}/\mathbf{U}(1)$ is the projective complex space $\mathbb{CP}^n$ and the projection map $\pi : \mathbb{S}^{2n+1} \rightarrow \mathbb{CP}^n$ is a Riemannian submersion with totally geodesic fibers isometric to $\mathbf{U}(1)$.

2.3.3 The Anti-de Sitter space

Let us consider the odd dimensional unit Lorentz sphere

$$\mathbf{Ad}_{2n+1} = \{z = (z_1, \cdots, z_{n+1}) \in \mathbb{C}^{n+1}, q(z) = 1\},$$

where q is the Lorentz norm

$$q(z) = \sum_{i=1}^{n} |z_i|^2 - |z_{n+1}|^2$$

There is an isometric group action of $\mathbb{S}^1 = \mathbf{U}(1)$ on $\mathbf{Ad}_{2n+1}$ which is defined by

$$(z_1, \cdots, z_n) \rightarrow (e^{i\theta} z_1, \cdots, e^{i\theta} z_n).$$

The quotient space $\mathbf{Ad}_{2n+1}/\mathbf{U}(1)$ is the complex hyperbolic space $\mathbb{CH}^n$ and the projection map $\pi : \mathbf{Ad}_{2n+1} \rightarrow \mathbb{CH}^n$ is a pseudo-Riemannian submersion with totally geodesic fibers isometric to $\mathbf{U}(1)$.

2.3.4 Sasakian spaces

Sasakian spaces are a special case of sub-Riemannian manifolds that arise in contact geometry.
Let $(\mathbb{M}, \theta)$ be a $2n + 1$-dimensional smooth contact manifold. On $\mathbb{M}$ there is a unique smooth vector field S, the so-called Reeb vector field, that satisfies

$$\theta(S) = 1, \quad \mathcal{L}_S(\theta) = 0,$$

where $\mathcal{L}_S$ denotes the Lie derivative with respect to S. On $\mathbb{M}$ there is a foliation, the Reeb foliation, whose leaves are the orbits of the vector field R. As it is well-known (see [51]), it is always possible to find a Riemannian metric g and a $(1,1)$-tensor field $\mathbf{J}$ on $\mathbb{M}$ so that for every horizontal vector fields X, Y

$$g(X, S) = \theta(X), \quad \mathbf{J}^2(X) = -X, \quad g(\mathbf{J}X, Y) = (d\theta)(X, Y), \quad \mathbf{J}S = 0.$$

The triple $(\mathbb{M}, \theta, g)$ is called a contact Riemannian manifold. It is then known and easy to prove that the Reeb foliation is totally geodesic with bundle like metric if and only if the Reeb vector field S is a Killing field, that is,

$$\mathcal{L}_S g = 0.$$

In that case, $(\mathbb{M}, \theta, g)$ is called a K-contact Riemannian manifold. Observe that the horizontal distribution $\mathcal{H}$ is then the kernel of θ and that $\mathcal{H}$ is bracket generating because θ is a contact form and therefore non degenerate. The Bott connection is the unique connection that satisfies:

1. $\nabla \theta = 0$;

2. $\nabla S = 0$;

3. $\nabla g = 0$;

4. $T(X, Y) = d\theta(X, Y)S$ for any $X, Y \in \Gamma^\infty(\mathcal{H})$;

5. $T(Z, X) = 0$ for any vector field $X \in \Gamma^\infty(\mathcal{H})$.

If $\mathbb{M}$ is a strongly pseudo-convex CR manifold with contact form θ, then the Tanno's connection is the Tanaka-Webster connection. In that case, we have then $\nabla \mathbf{J} = 0$ (see [23]). CR manifold of K-contact type are called Sasakian manifolds (see [23]).

In the following, we consider a complete Sasakian space $(\mathbb{M}, \theta, g)$. The sub-Riemannian structure of interest is $(\mathbb{M}, \mathcal{H}, g_\mathcal{H})$ where $\mathcal{H}$ is the kernel of θ. Observe that $\mathcal{H}$ is bracket-generating since θ is a contact form.

The Riemannian gradient will be denoted ∇ and we write the horizontal gradient as $\nabla_\mathcal{H}$, which is the projection of ∇ onto $\mathcal{H}$. Likewise, $\nabla_\mathcal{V}$ will denote the vertical gradient. The horizontal Laplacian (our sub-Laplacian) $\Delta_\mathcal{H}$ is the generator of the symmetric closable bilinear form:

$$\mathcal{E}_\mathcal{H}(f, g) = -\int_\mathbb{M} \langle \nabla_\mathcal{H} f, \nabla_\mathcal{H} g \rangle_\mathcal{H} \, d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

It is then easy to prove that:

$$\Delta = S^2 + \Delta_\mathcal{H}$$

where Δ is the Laplace-Beltrami operator of the Riemannian manifold $(\mathbb{M}, g)$. The operator S^2 can be understood as a vertical Laplacian, and we will often denote $\Delta_\mathcal{V} = S^2$. For a fixed $x_0 \in \mathbb{M}$ denote

$$r_0(x) = d_0(x_0, x).$$

The cut-locus $\mathbf{Cut}_0(x_0)$ of x_0 for the sub-Riemannian distance d_0 is defined as the complement of the set of y's in $\mathbb{M}$ such that there exists a unique length minimizing normal geodesic joining x_0 and y and x_0 and y are not conjugate along such geodesic (see [1]). The global cut-locus of $\mathbb{M}$ is defined by

$$\mathbf{Cut}_0(\mathbb{M}) = \{(x, y) \in \mathbb{M} \times \mathbb{M}, y \in \mathbf{Cut}_0(x)\}.$$

We first collect several known regularity results about r_0 in our Sasakian manifold framework.

Lemma 2.2 ([1], [37], [43])**.** *The following statements hold:*

1. *The distance function $x \to r_0(x)$ is locally semi-concave in $\mathbb{M} - \{x_0\}$. In particular, it is twice differentiable almost everywhere.*

2. *For any point x for which the function $x \to r_0(x)$ is differentiable, there exists a unique length minimizing sub-Riemannian geodesic and this geodesic is normal.*

3. *The set $\mathbb{M} - \mathbf{Cut}_0(x_0)$ is open and dense in $\mathbb{M}$.*

4. *The function $(x, y) \to d_0(x, y)^2$ is smooth on $\mathbb{M} \times \mathbb{M} - \mathbf{Cut}_0(\mathbb{M})$.*

To state the main theorem we are interested to prove, we start by introducing some helpful functions,

$$
\phi_\mu(t) = \begin{cases} \frac{\sinh \sqrt{\mu} t}{\sqrt{\mu}} & \text{if } \mu > 0, \\ t & \text{if } \mu = 0, \\ \frac{\sin \sqrt{|\mu|} t}{\sqrt{|\mu|}} & \text{if } \mu < 0, \end{cases} \qquad \psi_\mu(t) = \begin{cases} \frac{\sinh \sqrt{\mu} t - \sqrt{\mu} t}{\mu^{3/2}} & \text{if } \mu > 0, \\ \frac{1}{6} t^3 & \text{if } \mu = 0, \\ \frac{\sqrt{|\mu|} t - \sin \sqrt{|\mu|} t}{|\mu|^{3/2}} & \text{if } \mu < 0. \end{cases}
$$

Notice that $\psi_\mu(t) = \int_0^t \int_0^{s_2} \phi_\mu(s_1)\, ds_1\, ds_2$. We also introduce the following function:

$$
\Psi_\mu(r) = \begin{cases} \frac{1}{\mu^{3/2}}\left(\sqrt{\mu} - \frac{1}{r} \tanh \sqrt{\mu} r\right) & \text{if } \mu > 0, \\ \frac{1}{3} r^2 & \text{if } \mu = 0, \\ \frac{1}{|\mu|^{3/2}}\left(\frac{1}{r} \tan \sqrt{|\mu|} r - \sqrt{|\mu|}\right) & \text{if } \mu < 0. \end{cases}
$$

In the following, R will denote the full Riemannian curvature tensor of the Bott connection.

Theorem 2.3 (Horizontal Laplacian comparison theorem). *Let $k_1, k_2 \in \mathbb{R}$ and $\varepsilon > 0$. Assume that for every $X \in \Gamma^\infty(\mathcal{H})$, $\|X\| = 1$,*

$$
\langle R(X, \mathbf{J}X)\mathbf{J}X, X\rangle_{\mathcal{H}} \geq k_1, \tag{2.3}
$$

and that,

$$
\mathbf{Ric}_{\mathcal{H}}(X, X) - \langle R(X, \mathbf{J}X)\mathbf{J}X, X\rangle_{\mathcal{H}} \geq (n - 2)k_2. \tag{2.4}
$$

Let $x \neq x_0$ which is not in the cut-locus of x_0.

$$
\Delta_{\mathcal{H}} r_0 \leq (n - 2)\frac{\phi'_{-k_2}(r_0)}{\phi_{-k_2}(r_0)} + \frac{\phi'_{-k_1}(r_0)}{\phi_{-k_1}(r_0)} \frac{\Psi_{-k_1}(r_0)}{\Psi_{-k_1}(r_0/2)}.
$$

One deduces from the previous theorem that:

$$
\Delta_{\mathcal{H}} r_0 \leq \begin{cases} (n - 2)\sqrt{k_2}\cot\sqrt{k_2} r_0 + \left(\frac{\tan\sqrt{k_1} r_0 - r_0\sqrt{k_1}}{2\tan(\sqrt{k_1} r_0/2) - r_0\sqrt{k_1}}\right)\sqrt{k_1}\cot\sqrt{k_1} r_0 & \text{if } k_1, k_2 > 0, \\[2em] \frac{n+2}{r_0} & \text{if } k_1, k_2 = 0, \\[2em] (n - 2)\sqrt{|k_2|}\coth\sqrt{|k_2|} r_0 + \left(\frac{\sqrt{|k_1|} r_0 - \tanh\sqrt{|k_1|} r_0}{\sqrt{|k_1|} r_0 - 2\tanh(\sqrt{|k_1|} r_0/2)}\right)\sqrt{|k_1|}\coth\sqrt{|k_1|} r_0 & \text{if } k_1, k_2 < 0. \end{cases}
$$

Our theorem may be compared to Theorem 1.3 in [36] which holds under the same assumptions. There are however a few differences. The first one is that, for the sake of simplicity, we omit in our computations the terms involving the vertical derivative of the distance function, since this term is anyhow difficult to estimate. The second one is that we actually get a sharper constant for the term $(n-2)\frac{\phi'_{-k_2}(r_0)}{\phi_{-k_2}(r_0)}$ whereas [36] obtains the less precise $(n-1)\frac{\phi'_{-k_2}(r_0)}{\phi_{-k_2}(r_0)}$. As can be seen from [2], in the 3-dimensional case, Theorem 2.3 is optimal on the set of x for which $\nabla_{\mathcal{V}} r_0(x) = 0$ (worst possible situation). In Sasakian manifolds the quantity $K_{\mathbf{J}}(X) := \langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}}$ is called the pseudo-Hermitian sectional curvature of the Sasakian manifold (see [10] for a geometric interpretation). It can be seen as the CR analog of the holomorphic sectional curvature of a Kähler manifold. It is known that the holomorphic sectional curvature determines the whole curvature tensor, however there exist explicit examples of manifolds with positive holomorphic sectional curvature without any metric of positive Ricci curvature (see [30]). As a consequence it is likely that there exist examples for which k_1 and k_2 do not have the same sign.

Theorem 2.3 will be proved when $k_1 = k_2 = 0$ in the next sections. We first point out a relatively straightforward corollary

Theorem 2.4 (Sub-Riemannian Bonnet-Myers theorems). *Let $k_1, k_2 \in \mathbb{R}$ and $\varepsilon > 0$. Assume that for every $X \in \Gamma^\infty(\mathcal{H})$, $\|X\| = 1$,*

$$\langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}} \geq k_1, \tag{2.5}$$

and that,

$$\mathbf{Ric}_{\mathcal{H}}(X, X) - \langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}} \geq (n-2)k_2. \tag{2.6}$$

1. *If $k_1 > 0$, then $\mathbb{M}$ is compact and*

$$\mathbf{diam}\,(\mathbb{M}, d_0) \leq \frac{2\pi}{\sqrt{k_1}}.$$

2. *If $k_2 > 0$, then $\mathbb{M}$ is compact and*

$$\mathbf{diam}\,(\mathbb{M}, d_0) \leq \frac{\pi}{\sqrt{k_2}}.$$

2.4   Canonical variation of the distance

Our strategy to prove the sub-Laplacian comparison theorem is the following. The Riemannian metric g of the Sasakian space can be split as

$$g = g_{\mathcal{H}} \oplus g_{\mathcal{V}}, \tag{2.7}$$

9

and we introduce the one-parameter family of rescaled Riemannian metrics:

$$g_\varepsilon = g_\mathcal{H} \oplus \frac{1}{\varepsilon} g_\mathcal{V}, \quad \varepsilon > 0. \tag{2.8}$$

It is called the canonical variation of g (see [20], Chapter 9, for a discussion in the submersion case). The Riemannian distance associated with g_ε will be denoted by d_ε. It should be noted that d_ε, $\varepsilon > 0$, form an increasing (as $\epsilon \downarrow 0$) family of distances converging pointwise to the sub-Riemannian distance d_0.

To prove Theorem 2.3, the main strategy is to prove an estimate for $\Delta_\mathcal{H} r_\varepsilon$ by using the classical theory of Jacobi fields on the Riemannian manifolds $(\mathbb{M}, g_\varepsilon)$ and then take the limit $\varepsilon \to 0$.

The following lemma will be useful to take the limit:

Lemma 2.5. *Let $x \in \mathbb{M}$, $x \neq x_0$ which is not in $\cup_{n \geq 1} \mathbf{Cut}_{1/n}(x_0)$, then*

$$\lim_{n \to +\infty} \|\nabla_\mathcal{H} r_{1/n}(x)\|_g = 1.$$

Proof. Let $\gamma_n : [0,1] \to \mathbb{M}$ be the unique, constant speed, and length minimizing $g_{1/n}$ geodesic connecting x_0 to x. One has $d_{1/n}(x_0, x) \|\nabla_\mathcal{H} r_{1/n}(x)\|_g = \|\gamma_n'(0)\|_\mathcal{H}$. We therefore need to prove that $\lim_{n \to \infty} \|\gamma_n'(0)\|_\mathcal{H} = d_0(x_0, x)$. Let us observe that

$$\|\gamma_n'(0)\|_\mathcal{H}^2 + n\|\gamma_n'(0)\|_\mathcal{V}^2 = d_{1/n}(x_0, x)^2.$$

Therefore, $\lim_{n \to \infty} \|\gamma_n'(0)\|_\mathcal{V}^2 = 0$. Let us now assume that $\|\gamma_n'(0)\|_\mathcal{H}$ does not converge to $d_0(x_0, x)$. In that case, there exists a subsequence n_k such that $\|\gamma_{n_k}'(0)\|_\mathcal{H}$ converges to some $0 \leq a < d_0(x_0, x)$. For $f \in C_0^\infty(\mathbb{M})$ and $0 \leq s \leq t \leq 1$, we have

$$|f(\gamma_{n_k}(t)) - f(\gamma_{n_k}(s))| \leq \left(\|\gamma_{n_k}'(0)\|_\mathcal{H} \|\nabla_\mathcal{H} f\|_\infty + \|\gamma_{n_k}'(0)\|_\mathcal{V} \|\nabla_\mathcal{V} f\|_\infty \right) (t - s).$$

From Arzelà-Ascoli's theorem we deduce that there exists a subsequence which we continue to denote γ_{n_k} that converges uniformly to an absolutely continuous curve γ, such that $\gamma(0) = x_0$, $\gamma(1) = x$. We have for $f \in C_0^\infty(\mathbb{M})$ and $0 \leq s \leq t \leq 1$,

$$|f(\gamma(t)) - f(\gamma(s))| \leq a\|\nabla_\mathcal{H} f\|_\infty (t - s).$$

In particular, we deduce that

$$|f(x) - f(x_0)| \leq a\|\nabla_\mathcal{H} f\|_\infty.$$

Since it holds for every $f \in C_0^\infty(\mathbb{M})$, one deduces

$$d_0(x_0, x) = \sup\{|f(x) - f(x_0)|, \ f \in C_0^\infty(\mathbb{M}), \ \|\nabla_\mathcal{H} f\|_\infty \leq 1\} \leq a.$$

This contradicts the fact that $a < d_0(x_0, x)$. $\qquad\square$

2.5   Interlude: Second variation formulas and index forms

In this interlude, for the sake of reference, we collect without proofs several formulas used in the text. The main point is that the classical theory of second variations and Jacobi fields (see [22]) can be reformulated by using a connection which is not necessarily the Levi-Civita connection. To make the formulas and computations as straightforward and elegant as for the Levi-Civita connection, the only requirement is that we have to work with a metric connection whose adjoint is also metric.

Let $(\mathbb{M}, g)$ be a complete Riemannian manifold and ∇ be an affine metric connection on $\mathbb{M}$. We denote by $\hat{\nabla}$ the adjoint connection of ∇ given by

$$\hat{\nabla}_X Y = \nabla_X Y - T(X, Y),$$

where T is the torsion tensor of ∇. We will assume that $\hat{\nabla}$ is a metric connection. This is obviously equivalent to the fact that for every smooth vector fields X, Y, Z on $\mathbb{M}$, one has

$$\langle T(X, Y), Z \rangle = -\langle T(X, Z), Y \rangle. \tag{2.9}$$

Observe that the connection $(\nabla + \hat{\nabla})/2$ is torsion free and metric, it is therefore the Levi-Civita connection of the metric g. Let $\gamma : [0, T] \to \mathbb{M}$ be a smooth path on $\mathbb{M}$. The energy of γ is defined as

$$E(\gamma) = \frac{1}{2} \int_0^T \|\gamma'(t)\|^2 \, dt.$$

Let now X be a smooth vector field on γ with vanishing endpoints. One considers the variation of curves $\gamma(s, t) = \exp_{\gamma(t)}^{\nabla}(sX(\gamma(t)))$ where $\exp^{\nabla}$ is the exponential map of the connection ∇. The first variation of the energy $E(\gamma)$ is given by the formula:

$$\int_0^T \langle \gamma', \nabla_\gamma' X + T(X, \gamma') \rangle \, dt = \int_0^T \langle \gamma', \hat{\nabla}_\gamma' X \rangle \, dt = -\int_0^T \langle \hat{\nabla}_\gamma' \gamma', X \rangle \, dt.$$

As a consequence, the critical curves of E are the geodesics of the adjoint connection $\hat{\nabla}$:

$$\hat{\nabla}_\gamma' \gamma' = 0.$$

These critical curves are also geodesics for ∇ and for the Levi-Civita connection and thus distance minimizing if the endpoints are not in the cut-locus. One can also compute the second variation of the energy at a geodesic γ and standard computations yield

$$\int_0^T \left(\langle \nabla_{\gamma'} X, \hat{\nabla}_{\gamma'} X \rangle - \langle \hat{R}(\gamma', X)X, \gamma' \rangle \right) dt \tag{2.10}$$

where $\hat{R}$ is the Riemann curvature tensor of $\hat{\nabla}$. This is the formula for the second variation with fixed endpoints. This formula does not depend on the choice of connection ∇.

The index form of a vector field X (with not necessarily vanishing endpoints) along a geodesic γ is given by

$$I(\gamma, X, X) := \int_0^T \left(\langle \nabla_{\gamma'} X, \hat{\nabla}_{\gamma'} X \rangle - \langle \hat{R}(\gamma', X)X, \gamma' \rangle \right) dt$$

$$= \int_0^T \left(\langle \nabla_{\gamma'} X, \hat{\nabla}_{\gamma'} X \rangle - \langle R(\gamma', X)X, \gamma' \rangle \right) dt.$$

If Y is a Jacobi field along the geodesic γ, one has

$$\hat{\nabla}_{\gamma'} \nabla_{\gamma'} Y = \hat{\nabla}_{\gamma'} \hat{\nabla}_Y \gamma' = \hat{R}(\gamma', Y)\gamma'$$

because $\hat{\nabla}_{\gamma'} \gamma' = 0$. The Jacobi equation therefore writes

$$\hat{\nabla}_{\gamma'} \nabla_{\gamma'} Y = \hat{R}(\gamma', Y)\gamma'. \tag{2.11}$$

We have then the following results:

Lemma 2.6. *Let $x_0 \in \mathbb{M}$ and $x \neq x_0$ which is not in the cut-locus of x. We denote by $r = d(x_0, \cdot)$ the distance function from x_0. Let $X \in T_x\mathbb{M}$ be orthogonal to $\nabla r(x)$. At the point x, we have*

$$\nabla^2 r(X, X) = I(\gamma, Y, Y)$$

where γ is the unique length parametrized geodesic connecting x_0 to x and Y the Jacobi field along γ such that $Y(0) = 0$ and $Y(r(x)) = X$.

Combining this with the index lemma yields:

Lemma 2.7. *Let $x_0 \in \mathbb{M}$ and $x \neq x_0$ which is not in the cut-locus of x. Let $X \in T_x\mathbb{M}$. At x, we have*

$$\nabla^2 r(X, X) \leq \int_0^r \left(\langle \nabla_{\gamma'} \tilde{X}, \hat{\nabla}_{\gamma'} \tilde{X} \rangle - \langle \hat{R}(\gamma', \tilde{X}) \tilde{X}, \gamma' \rangle \right) dt$$

where γ is the unique length parametrized geodesic connecting x_0 to x and $\tilde{X}$ is any vector field along γ such that $\tilde{X}(0) = 0$ and $\tilde{X}(r(x)) = X$.

2.6 Horizontal Hessian and Laplacian comparison theorems in non-negative curvature

In this section, we prove a Hessian comparison theorem for r_ε under the assumption of non-negative sectional curvature and deduce a horizontal Laplacian comparison theorem. The idea is to use a convenient family of *almost* Jacobi fields, yielding optimal results when $\varepsilon \to 0$. The result shall be generalized to negative and positive sectional curvature in the next sections, but since the computations are quite involved, for the sake of presentation, we first present the case of non-negative sectional curvature. Our main theorem is the following:

Theorem 2.8. *Assume that the horizontal sectional curvature of the Bott connection is non negative, namely for every horizontal fields X, Y,*

$$\langle R(X, Y)Y, X \rangle_{\mathcal{H}} \geq 0.$$

Let $x \neq x_0$ which is not in the cut-locus of x_0. Let $X \in T_x\mathbb{M}$ which is horizontal, orthogonal to $\nabla_{\mathcal{H}} r_\varepsilon(x)$ and such that $\|X\|_{\mathcal{H}} = 1$. One has at x

$$\nabla^2_{\mathcal{H}} r_\varepsilon(X, X) \leq \frac{1}{r_\varepsilon} + \frac{1}{4\varepsilon} \frac{r_\varepsilon \langle \mathbf{J}X, \nabla_{\mathcal{H}} r_\varepsilon \rangle^2_{\mathcal{H}}}{1 + \frac{\|\nabla_{\mathcal{H}} r_\varepsilon\|^2 r_\varepsilon^2}{12\varepsilon}}.$$

Proof. Let γ be the unique length parametrized geodesic connecting x_0 to x. We consider at x the vertical vector

$$Z = \frac{1}{2} \frac{T(X, \gamma')}{1 + \frac{r_\varepsilon^2 \|\gamma'\|^2_{\mathcal{H}}}{12\varepsilon}}.$$

We still denote by Z the vector field along γ which is obtained by parallel transport of Z for the Bott connection ∇. We will also still denote by X the vector field along γ which is obtained by parallel transport of $X \in T_x\mathbb{M}$ for the adjoint connection $\hat{\nabla}^\varepsilon = \nabla + \frac{1}{\varepsilon} J$. We now consider the vector field Y defined along γ by:

$$Y(t) = -\frac{1}{2\varepsilon} t(t - r_\varepsilon) J_Z \gamma' + \frac{t}{r_\varepsilon} X + \left(t - \frac{1}{2\varepsilon} \left(\frac{t^3}{3} - \frac{1}{2} r_\varepsilon t^2 \right) \|\gamma'\|^2_{\mathcal{H}} \right) Z + \frac{t^2}{2r_\varepsilon} T(\gamma', X).$$

We observe that:

1. $Y(0) = 0$ and $Y(r_\varepsilon) = X$;

2. $\nabla_{\gamma'} Y_{\mathcal{V}} = T(\gamma', Y) + Z$;

3. $\hat{\nabla}^\varepsilon_{\gamma'} Y_{\mathcal{H}} = \frac{1}{r_\varepsilon} X - \frac{1}{2\varepsilon}(2t - \varepsilon) J_Z \gamma'$.

For 2., we used the fact that since $\mathbb{M}$ is a Sasakian manifold $T(\gamma', J_Z \gamma') = \|\gamma'\|^2_{\mathcal{H}} Z$. We point out that Y is actually the solution of the differential equation:

$$\nabla^\varepsilon_{\gamma'} \hat{\nabla}^\varepsilon_{\gamma'} Y + \frac{1}{\varepsilon} J_{\gamma'} \hat{\nabla}^\varepsilon_{\gamma'} Y = \frac{1}{\varepsilon} J_{T(\gamma', Y)} \gamma'$$

with the boundary conditions $Y(0) = 0$, $Y(r_\varepsilon) = X$. Thus, even when $R = 0$ (Heisenberg group case), Y is not a Jacobi field, unless γ is horizontal. However, considering this Y simplifies computations, and moreover for $\varepsilon \to 0$, we shall get optimal results since γ will converge to a sub-Riemannian geodesic, which is horizontal.

From lemma 2.6 and the index lemma, one has

$$\nabla^2_{\mathcal{H}} r_\varepsilon(X, X) \leq \int_0^{r_\varepsilon} \langle \nabla^\varepsilon_{\gamma'} Y, \hat{\nabla}^\varepsilon_{\gamma'} Y \rangle_\varepsilon - \langle \hat{R}^\varepsilon(\gamma', Y)Y, \gamma' \rangle_\varepsilon.$$

13

We now observe that

$$\langle \hat{R}^\varepsilon(\gamma', Y)Y, \gamma'\rangle_\varepsilon = \langle R(\gamma', Y)Y, \gamma'\rangle_\varepsilon - \|T(Y, \gamma')\|_\varepsilon^2$$
$$= \langle R(\gamma'_\mathcal{H}, Y_\mathcal{H})Y_\mathcal{H}, \gamma'_\mathcal{H}\rangle_\mathcal{H} - \|T(Y, \gamma')\|_\varepsilon^2$$
$$\geq -\|T(Y, \gamma')\|_\varepsilon^2.$$

Therefore we have

$$\nabla_\mathcal{H}^2 r_\varepsilon(X, X) \leq \int_0^{r_\varepsilon} \langle \nabla_{\gamma'}^\varepsilon Y, \hat{\nabla}_{\gamma'}^\varepsilon Y\rangle_\varepsilon + \|T(Y, \gamma')\|_\varepsilon^2.$$

A lengthy but routine computation yields

$$\int_0^{r_\varepsilon} \langle \nabla_{\gamma'}^\varepsilon Y, \hat{\nabla}_{\gamma'}^\varepsilon Y\rangle_\varepsilon + \|T(Y, \gamma')\|_\varepsilon^2$$
$$= \frac{1}{r_\varepsilon} + \left(\frac{r_\varepsilon}{\varepsilon} + \frac{r_\varepsilon^3}{12\varepsilon^2}\|\nabla_\mathcal{H} r_\varepsilon\|_\mathcal{H}^2\right)\|Z\|_\varepsilon^2 + \frac{1}{2\varepsilon}\|T(X, \gamma')\|_\varepsilon^2 + \left(\frac{r_\varepsilon}{\varepsilon} + \frac{r_\varepsilon^3}{12\varepsilon^2}\|\nabla_\mathcal{H} r_\varepsilon\|_\mathcal{H}^2\right)\langle Z, T(\gamma', X)\rangle_\varepsilon.$$

Using the fact that

$$Z = \frac{1}{2}\frac{T(X, \gamma')}{1 + \frac{r_\varepsilon^2\|\gamma'\|_\mathcal{H}^2}{12\varepsilon}},$$

one gets

$$\int_0^{r_\varepsilon} \langle \nabla_{\gamma'}^\varepsilon Y, \hat{\nabla}_{\gamma'}^\varepsilon Y\rangle_\varepsilon + \|T(Y, \gamma')\|_\varepsilon^2 = \frac{1}{r_\varepsilon} + \frac{1}{4\varepsilon}\frac{r_\varepsilon\langle \mathbf{J}X, \nabla_\mathcal{H} r_\varepsilon\rangle_\mathcal{H}^2}{1 + \frac{\|\nabla_\mathcal{H} r_\varepsilon\|^2 r_\varepsilon^2}{12\varepsilon}}.$$

The proof is then complete. $\qquad\square$

We now estimate $\nabla_\mathcal{H}^2 r_\varepsilon(\nabla_\mathcal{H} r_\varepsilon, \nabla_\mathcal{H} r_\varepsilon)$, which was the missing direction in the previous theorems.

Lemma 2.9. *Let $x \neq x_0$ which is not in the cut-locus of x_0. We have at x:*

$$\nabla_\mathcal{H}^2 r_\varepsilon(\nabla_\mathcal{H} r_\varepsilon, \nabla_\mathcal{H} r_\varepsilon) \leq \frac{\min(\Gamma(r_\varepsilon), 1 - \Gamma(r_\varepsilon))}{r_\varepsilon}.$$

Proof. From the index lemma, one has

$$\nabla_\mathcal{H}^2 r_\varepsilon(\nabla_\mathcal{H} r_\varepsilon, \nabla_\mathcal{H} r_\varepsilon) \leq I_{\nabla^\varepsilon}(\gamma, X, X),$$

where γ is the unique length parametrized geodesic connecting x_0 to x and $X = \frac{t}{r_\varepsilon}\gamma'_\mathcal{H}$. An immediate computation gives

$$I_{\nabla^\varepsilon}(\gamma, X, X) = \frac{\Gamma(r_\varepsilon)}{r_\varepsilon}.$$

Therefore

$$\nabla_\mathcal{H}^2 r_\varepsilon(\nabla_\mathcal{H} r_\varepsilon, \nabla_\mathcal{H} r_\varepsilon) \leq \frac{\Gamma(r_\varepsilon)}{r_\varepsilon}.$$

14

We now observe that

$$\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2 + \varepsilon \|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2 = 1.$$

As a consequence

$$\nabla_{\mathcal{H}} \|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2 + \varepsilon \nabla_{\mathcal{H}} \|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2 = 0$$

and

$$\nabla_{\mathcal{V}} \|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2 + \varepsilon \nabla_{\mathcal{V}} \|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2 = 0.$$

From the first equality we deduce

$$\langle \nabla_{\mathcal{H}} \|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2, \nabla_{\mathcal{H}} r_\varepsilon \rangle_{\mathcal{H}} + \varepsilon \langle \nabla_{\mathcal{H}} \|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2, \nabla_{\mathcal{H}} r_\varepsilon \rangle = 0.$$

and therefore,

$$\nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) + \varepsilon \nabla_{\mathcal{H},\mathcal{V}}^2 r_\varepsilon(\nabla r_\varepsilon, \nabla r_\varepsilon) = 0.$$

Similarly, from the second equality we have

$$\nabla_{\mathcal{V},\mathcal{H}}^2 r_\varepsilon(\nabla r_\varepsilon, \nabla r_\varepsilon) + \varepsilon \nabla_{\mathcal{V}}^2 r_\varepsilon(\nabla_{\mathcal{V}} r_\varepsilon, \nabla_{\mathcal{V}} r_\varepsilon) = 0.$$

It is easy to check that $\nabla_{\mathcal{V},\mathcal{H}}^2 = \nabla_{\mathcal{H},\mathcal{V}}^2$ (see [14]). Consequently,

$$\nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) = \varepsilon^2 \nabla_{\mathcal{V}}^2 r_\varepsilon(\nabla_{\mathcal{V}} r_\varepsilon, \nabla_{\mathcal{V}} r_\varepsilon) = \varepsilon^2 \|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2 \Delta_{\mathcal{V}} r_\varepsilon.$$

From a vertical Laplacian comparison theorem one has

$$\Delta_{\mathcal{V}} r_\varepsilon \leq \frac{1}{\varepsilon r_\varepsilon}.$$

This yields

$$\nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) \leq \varepsilon \frac{\|\nabla_{\mathcal{V}} r_\varepsilon\|_{\mathcal{V}}^2}{r_\varepsilon} = \frac{1 - \|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2}{r_\varepsilon} = \frac{1 - \Gamma(r_\varepsilon)}{r_\varepsilon}.$$

$$\square$$

We now prove the horizontal Laplacian theorem:

Theorem 2.10. *Assume that for every* $X \in \Gamma^\infty(\mathcal{H})$, $\|X\| = 1$,

$$\mathbf{Ric}_{\mathcal{H}}(X, X) \geq \langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}} \geq 0.$$

Let $x \neq x_0$ *which is not in the cut-locus of* x_0. *At* x *we have*

$$\Delta_{\mathcal{H}} r_\varepsilon \leq \frac{n+2}{r_\varepsilon} + \frac{1}{r_\varepsilon} \min\left(\frac{1}{\Gamma(r_\varepsilon)} - 1, 1\right).$$

15

Proof. We first assume that $\nabla_{\mathcal{H}} r_\varepsilon(x) \neq 0$. Let $X_1, \cdots, X_{n-2}$ be horizontal vectors at x such that $\left(X_1, \cdots, X_{n-2}, \frac{\nabla_{\mathcal{H}} r_\varepsilon}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}}, \frac{\mathbf{J}\nabla_{\mathcal{H}} r_\varepsilon}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}}\right)$ is a horizontal orthonormal frame. From the proof of theorem 2.8, we get

$$\Delta_{\mathcal{H}} r_\varepsilon = \frac{1}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2} \nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) + \frac{1}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2} \nabla_{\mathcal{H}}^2 r_\varepsilon(\mathbf{J}\nabla_{\mathcal{H}} r_\varepsilon, \mathbf{J}\nabla_{\mathcal{H}} r_\varepsilon) + \sum_{i=1}^{n-2} \nabla_{\mathcal{H}}^2 r_\varepsilon(X_i, X_i)$$

$$\leq \frac{1}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2} \nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) + \frac{n-1}{r_\varepsilon} + \frac{3}{r_\varepsilon}$$

$$\leq \frac{1}{\|\nabla_{\mathcal{H}} r_\varepsilon\|_{\mathcal{H}}^2} \nabla_{\mathcal{H}}^2 r_\varepsilon(\nabla_{\mathcal{H}} r_\varepsilon, \nabla_{\mathcal{H}} r_\varepsilon) + \frac{n+2}{r_\varepsilon}$$

$$\leq \frac{n+2}{r_\varepsilon} + \frac{1}{r_\varepsilon} \min\left(\frac{1}{\Gamma(r_\varepsilon)} - 1, 1\right).$$

If $\nabla_{\mathcal{H}} r_\varepsilon(x) = 0$, one can find $X_1, \cdots, X_n$ horizontal vectors at x such that $(X_1, \cdots, X_n)$ is a horizontal orthonormal frame and theorem 2.8 yields

$$\Delta_{\mathcal{H}} r_\varepsilon \leq \frac{n+3}{r_\varepsilon}.$$

$\square$

To finish the section, we study the limit when $\varepsilon \to 0$ in the previous theorem.
We can now conclude:

Theorem 2.11. *Assume that for every* $X \in \Gamma^\infty(\mathcal{H})$, $\|X\| = 1$,

$$\langle R(X, \mathbf{J}X)\mathbf{J}X, X\rangle_{\mathcal{H}} \geq 0$$

$$\mathbf{Ric}_{\mathcal{H}}(X, X) - \langle R(X, \mathbf{J}X)\mathbf{J}X, X\rangle_{\mathcal{H}} \geq 0.$$

Then, outside of $\mathbf{Cut}_0(x_0)$, *and in the sense of distribution, for the sub-Riemannian distance we have*

$$\Delta_{\mathcal{H}} r_0 \leq \frac{n+2}{r_0}.$$

Proof. Since the cut-locus of x_0 for the metric g_ε has measure 0, by usual arguments, we have in the sense of distributions:

$$\Delta_{\mathcal{H}} r_\varepsilon \leq \frac{n+2}{r_\varepsilon} + \frac{1}{r_\varepsilon} \min\left(\frac{1}{\Gamma(r_\varepsilon)} - 1, 1\right).$$

This means that for every smooth, non negative and compactly supported function f,

$$\int_{\mathbb{M}} (\Delta_{\mathcal{H}} f)\, r_\varepsilon d\mu \leq \int_{\mathbb{M}} \left(\frac{n+2}{r_\varepsilon} + \frac{1}{r_\varepsilon} \min\left(\frac{1}{\Gamma(r_\varepsilon)} - 1, 1\right)\right) f d\mu.$$

Taking the limit when $\varepsilon \to 0$ yields the result. $\square$

2.7 Measure contraction properties

We now recall the following definition (see [40, 49, 50]).

Definition 2.12. *Let (X, δ, ν) be a metric measure space. Assume that for every $x_0 \in X$ there exists a Borel set Ω_{x_0} of full measure in X (that is $\nu(X \setminus \Omega_{x_0}) = 0$) such that any point of Ω_{x_0} is connected to x_0 by a unique distance minimizing geodesic $t \to \phi_{t,x_0}(x)$, $t \in [0, 1]$, starting at x and ending at x_0. We say that (X, δ, ν) satisfies the measure contraction property $\mathbf{MCP}(0, N)$, $N \geq 0$, if for every $x_0 \in X$, $t \in [0, 1]$ and Borel set U,*

$$\nu(\phi_{t,x_0}(U)) \geq (1 - t)^N \nu(U).$$

As an immediate corollary of Theorem 2.10, we deduce:

Corollary 2.13. *Let $(\mathbb{M}, g_{\mathcal{H}})$ be a Sasakian structure such that*

$$\mathbf{K}_{\mathcal{H},J} \geq 0, \ \ \mathbf{Ric}_{\mathcal{H},J^\perp} \geq 0.$$

Then, for every $\varepsilon > 0$, the metric measure space $(\mathbb{M}, d_\varepsilon, \mu)$ satisfies the measure contraction property $\mathbf{MCP}(0, n + 5)$.

This corollary is interesting because the Ricci tensor of the metric g_ε for the Levi-Civita connection blows up to $-\infty$ in the directions of the horizontal space when $\varepsilon \to 0$. Such similar situations are pointed out in Lee [35].

Proposition 2.14. *Let $(\mathbb{M}, g_{\mathcal{H}})$ be a Sasakian structure such that for every $X \in \Gamma^\infty(\mathcal{H})$, $\|X\| = 1$,*

$$\langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}} \geq 0$$

$$\mathbf{Ric}_{\mathcal{H}}(X, X) - \langle R(X, \mathbf{J}X)\mathbf{J}X, X \rangle_{\mathcal{H}} \geq 0.$$

Then the metric measure space $(\mathbb{M}, d_0, \mu)$ satisfies the measure contraction property $\mathbf{MCP}$ $(0, n + 3)$.

3 Volume doubling properties on sub-Riemannian manifolds

3.1 Framework

Throughout the section, we consider a smooth connected $n + m$ dimensional manifold $\mathbb{M}$ which is equipped with a Riemannian foliation with a bundle like metric g and totally geodesic m dimensional leaves. We moreover assume that the metric g is complete and that the horizontal distribution $\mathcal{H}$ of the foliation is bracket-generating. We denote by μ the Riemannian reference volume measure on $\mathbb{M}$.

As is usual, the sub-bundle $\mathcal{V}$ formed by vectors tangent to the leaves is referred to as the set of *vertical directions*. The sub-bundle $\mathcal{H}$ which is normal to $\mathcal{V}$ is referred to as the set of *horizontal directions*. Saying that the foliation is totally geodesic means that:

$$(\mathcal{L}_X g)(Z, Z) = 0, \qquad (\mathcal{L}_Z g)(X, X) = 0, \qquad \text{for any } X \in \Gamma^\infty(\mathcal{H}), \ Z \in \Gamma^\infty(\mathcal{V}).$$

The literature on Riemannian foliations is vast, we refer for instance to the classical reference [53] and its bibliography for further details.

The Riemannian gradient will be denoted ∇ and we write the horizontal gradient as $\nabla_{\mathcal{H}}$, which is the projection of ∇ onto $\mathcal{H}$. Likewise, $\nabla_{\mathcal{V}}$ will denote the vertical gradient. The horizontal Laplacian $\Delta_{\mathcal{H}}$ is the generator of the symmetric closable bilinear form:

$$\mathcal{E}_{\mathcal{H}}(f, g) = -\int_{\mathbb{M}} \langle \nabla_{\mathcal{H}} f, \nabla_{\mathcal{H}} g \rangle_{\mathcal{H}} \, d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

The vertical Laplacian may be defined as $\Delta_{\mathcal{V}} = \Delta - \Delta_{\mathcal{H}}$ where Δ is the Laplace-Beltrami operator on $\mathbb{M}$. We have

$$\mathcal{E}_{\mathcal{V}}(f, g) := -\int_{\mathbb{M}} \langle \nabla_{\mathcal{V}} f, \nabla_{\mathcal{V}} g \rangle_{\mathcal{V}} \, d\mu = \int_{\mathbb{M}} f \Delta_{\mathcal{V}} g \, d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

The hypothesis that $\mathcal{H}$ is bracket generating implies that the horizontal Laplacian $\Delta_{\mathcal{H}}$ is locally subelliptic and the completeness assumption on g implies that $\Delta_{\mathcal{H}}$ is essentially self-adjoint on the space of smooth and compactly supported functions (see for instance [12]).

3.2 Bott connection

There is a first natural connection on $\mathbb{M}$ that respects the foliation structure, the Bott connection, which is given as follows:

$$\nabla_X Y = \begin{cases} \pi_{\mathcal{H}}(\nabla_X^g Y), & X, Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{H}}([X, Y]), & X \in \Gamma^\infty(\mathcal{V}), \ Y \in \Gamma^\infty(\mathcal{H}), \\ \pi_{\mathcal{V}}([X, Y]), & X \in \Gamma^\infty(\mathcal{H}), \ Y \in \Gamma^\infty(\mathcal{V}), \\ \pi_{\mathcal{V}}(\nabla_X^g Y), & X, Y \in \Gamma^\infty(\mathcal{V}), \end{cases}$$

where ∇^g is the Levi-Civita connection for g and $\pi_{\mathcal{H}}$ (resp. $\pi_{\mathcal{V}}$) the projection on $\mathcal{H}$ (resp. $\mathcal{V}$). It is easy to check that for every $\varepsilon > 0$, this connection satisfies $\nabla g_\varepsilon = 0$. A fundamental property of ∇ is that $\mathcal{H}$ and $\mathcal{V}$ are parallel.

The torsion T of ∇ is given as

$$T(X,Y) = -\pi_{\mathcal{V}}[\pi_{\mathcal{H}}X, \pi_{\mathcal{H}}Y].$$

For $Z \in \Gamma^\infty(\mathcal{V})$, there is a unique skew-symmetric endomorphism $J_Z : \mathcal{H}_x \to \mathcal{H}_x$ such that for all horizontal vector fields X and Y,

$$g_{\mathcal{H}}(J_Z(X), Y) = g_{\mathcal{V}}(Z, T(X,Y)), \tag{3.1}$$

where T is the torsion tensor of ∇. We extend J_Z to be 0 on $\mathcal{V}_x$. Also, if $Z \in \Gamma^\infty(\mathcal{H})$, from (3.1) we set $J_Z = 0$.

We now introduce the relevant tensors which will be used to control the index forms. The horizontal divergence of the torsion T is the $(1,1)$ tensor which in a local horizontal frame $X_1, \ldots, X_n$ is defined by

$$\delta_{\mathcal{H}}T(X) := -\sum_{j=1}^{n}(\nabla_{X_j}T)(X_j, X).$$

Going forward, we will always assume in the sequel of the course that the horizontal distribution $\mathcal{H}$ satisfies *the Yang-Mills condition*, meaning that $\delta_{\mathcal{H}}T = 0$ (see [12, 26, 27] for the geometric significance of this condition).

We will denote by $\mathbf{Ric}_{\mathcal{H}}$ the horizontal Ricci curvature of the Bott connection, that is to say the horizontal trace of the full curvature tensor R of the Bott connection. Using the observation that ∇ preserves the splitting $\mathcal{H} \oplus \mathcal{V}$ and from the first Bianchi identity, it follows that $\mathbf{Ric}_{\mathcal{H}}(X,Y) = \mathbf{Ric}_{\mathcal{H}}(\pi_{\mathcal{H}}X, \pi_{\mathcal{H}}Y)$. We will also denote $\mathbf{Ric}_{\mathcal{V}}$ the vertical Ricci curvature of the Bott connection (this is also the Ricci curvature of the leaves of the foliation as sub-manifolds of $(\mathbb{M}, g)$)

If $Z_1, \ldots, Z_m$ is a local vertical frame, the $(1,1)$ tensor

$$\mathbf{J}^2 := \sum_{\ell=1}^{m} J_{Z_\ell} J_{Z_\ell}$$

does not depend on the choice of the frame and may globally be defined.

3.3 Horizontal and vertical Bochner formulas and Laplacian comparison theorems

It is well-known that on Riemannian manifolds the Laplacian comparison theorem may also be obtained as a consequence of the Bochner formula.

We first recall the horizontal and vertical Bochner identities that were respectively proved in [19] and [14] (see also [26, 27] for generalizations going beyond the foliation case).

Theorem 3.1 (Horizontal and vertical Bochner identities). *For $f \in C^\infty(\mathbb{M})$, one has*

$$\frac{1}{2}\Delta_{\mathcal{H}}\|df\|_\varepsilon^2 - \langle d\Delta_{\mathcal{H}}f, df\rangle_\varepsilon = \|\nabla_{\mathcal{H}}^\varepsilon df\|_\varepsilon^2 + \langle \mathbf{Ric}_{\mathcal{H}}(df), df\rangle_{\mathcal{H}} + \frac{1}{\varepsilon}\langle \mathbf{J}^2(df), df\rangle_{\mathcal{H}},$$

and

$$\frac{1}{2}\Delta_{\mathcal{V}}\|df\|_\varepsilon^2 - \langle d\Delta_{\mathcal{V}}f, df\rangle_\varepsilon = \|\nabla_{\mathcal{H},\mathcal{V}}^2 f\|^2 + \varepsilon\|\nabla_{\mathcal{V}}^2 f\|^2 + \varepsilon\langle \mathbf{Ric}_{\mathcal{V}}(df), df\rangle_{\mathcal{V}}.$$

Proof. The first identity is Theorem 3.1 in [19]. The second identity may be derived from Proposition 2.2 in [14]. $\square$

Those two Bochner formulas may be used to prove general curvature dimension estimates respectively for the horizontal and vertical Laplacian.
We introduce the following operators defined for $f, g \in C^\infty(\mathbb{M})$,

$$\Gamma(f,g) = \frac{1}{2}(\Delta_{\mathcal{H}}(fg) - g\Delta_{\mathcal{H}}f - f\Delta_{\mathcal{H}}g) = \langle \nabla_{\mathcal{H}}f, \nabla_{\mathcal{H}}g\rangle_{\mathcal{H}},$$

$$\Gamma^{\mathcal{V}}(f,g) = \langle \nabla_{\mathcal{V}}f, \nabla_{\mathcal{V}}g\rangle_{\mathcal{V}},$$

and their iterations which are defined by

$$\Gamma_2^{\mathcal{H}}(f,g) = \frac{1}{2}(\Delta_{\mathcal{H}}(\Gamma(f,g)) - \Gamma(g, \Delta_{\mathcal{H}}f) - \Gamma(f, \Delta_{\mathcal{H}}g))$$

$$\Gamma_2^{\mathcal{H},\mathcal{V}}(f,g) = \frac{1}{2}(\Delta_{\mathcal{H}}(\Gamma^{\mathcal{V}}(f,g)) - \Gamma^{\mathcal{V}}(g, \Delta_{\mathcal{H}}f) - \Gamma^{\mathcal{V}}(f, \Delta_{\mathcal{H}}g))$$

$$\Gamma_2^{\mathcal{V},\mathcal{H}}(f,g) = \frac{1}{2}(\Delta_{\mathcal{V}}(\Gamma(f,g)) - \Gamma(g, \Delta_{\mathcal{V}}f) - \Gamma(f, \Delta_{\mathcal{V}}g))$$

and

$$\Gamma_2^{\mathcal{V}}(f,g) = \frac{1}{2}(\Delta_{\mathcal{V}}(\Gamma^{\mathcal{V}}(f,g)) - \Gamma^{\mathcal{V}}(g, \Delta_{\mathcal{V}}f) - \Gamma^{\mathcal{V}}(f, \Delta_{\mathcal{V}}g))$$

As a straightforward consequence of Theorem 3.1, we obtain the following generalized curvature dimension inequalities for the horizontal and vertical Laplacians.

Theorem 3.2.

1. *Assume that globally on $\mathbb{M}$, for every $X \in \Gamma^\infty(\mathcal{H})$ and $Z \in \Gamma^\infty(\mathcal{V})$,*

$$\mathbf{Ric}_{\mathcal{H}}(X,X) \geq \rho_1(r_\varepsilon)\|X\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2 X, X\rangle_{\mathcal{H}} \leq \kappa(r_\varepsilon)\|X\|_{\mathcal{H}}^2, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_Z^2) \geq \rho_2(r_\varepsilon)\|Z\|_{\mathcal{V}}^2,$$

for some continuous functions ρ_1, ρ_2, κ. For every $f \in C^\infty(\mathbb{M})$, one has

$$\Gamma_2^{\mathcal{H}}(f,f) + \varepsilon\Gamma_2^{\mathcal{H},\mathcal{V}}(f,f) \geq \frac{1}{n}(\Delta_{\mathcal{H}}f)^2 + \left(\rho_1(r_\varepsilon) - \frac{\kappa(r_\varepsilon)}{\varepsilon}\right)\Gamma(f,f) + \rho_2(r_\varepsilon)\Gamma^{\mathcal{V}}(f,f).$$

20

2. *Assume that globally on $\mathbb{M}$, for every $Z \in \Gamma^\infty(\mathcal{V})$,*

$$\mathbf{Ric}_\mathcal{V}(Z, Z) \geq \rho_3(r_\varepsilon)\|Z\|_\mathcal{V}^2,$$

for some continuous functions ρ_3. For every $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma_2^{\mathcal{V},\mathcal{H}}(f, f) + \varepsilon\Gamma_2^\mathcal{V}(f, f) \geq \frac{\varepsilon}{m}(\Delta_\mathcal{V}f)^2 + \varepsilon\rho_3(r_\varepsilon)\Gamma^\mathcal{V}(f, f).$$

Proof. The proof of 1. follows from

$$\|\nabla_\mathcal{H}^\varepsilon df\|_\varepsilon^2 \geq \|\nabla_\mathcal{H}^2 f\|^2 - \frac{1}{4}\mathbf{Tr}_\mathcal{H}(J_{df}^2)$$

$$\geq \frac{1}{n}(\Delta_\mathcal{H}f)^2 + \rho_2(r_\varepsilon)\Gamma^\mathcal{V}(f, f),$$

where we refer to the proof of Theorem 3.1 in [19] for the details. The proof of 2. is immediate. □

We deduce the following theorems:

Theorem 3.3. *Let $x \in \mathbb{M}$, $x \neq x_0$ and x not in the d_ε cut-locus of x_0. Let $G : [0, r_\varepsilon(x)] \to \mathbb{R}_{\geq 0}$ be a differentiable function which is positive on $(0, r_\varepsilon(x)]$ and such that $G(0) = 0$. We have*

$$\Delta_\mathcal{H}r_\varepsilon(x) \leq \frac{1}{G(r_\varepsilon(x))^2}\int_0^{r_\varepsilon(x)}\left(nG'(s)^2 - \left[\left(\rho_1(s) - \frac{1}{\varepsilon}\kappa(s)\right)\Gamma(r_\varepsilon)(x) + \rho_2(s)\Gamma^\mathcal{V}(r_\varepsilon)(x)\right]G(s)^2\right)ds.$$

Theorem 3.4. *Let $x \in \mathbb{M}$, $x \neq x_0$, not in the d_ε cut-locus of x_0. Let $G : [0, r_\varepsilon(x)] \to \mathbb{R}_{\geq 0}$ be a differentiable function which is positive on $(0, r_\varepsilon(x)]$ and such that $G(0) = 0$. We have*

$$\Delta_\mathcal{V}r_\varepsilon(x) \leq \frac{1}{G(r_\varepsilon(x))^2}\int_0^{r_\varepsilon(x)}\left(\frac{m}{\varepsilon}G'(s)^2 - \rho_3(s)\varepsilon\Gamma^\mathcal{V}(r_\varepsilon)(x)G(s)^2\right)ds.$$

To prove Theorem 3.3 and 3.4, we shall need the easily proved following lemma.

Lemma 3.5. *We have*

$$\lim_{x \to x_0} r_\varepsilon(x)^2\Delta_\mathcal{H}r_\varepsilon(x) = \lim_{x \to x_0} r_\varepsilon(x)^2\Delta_\mathcal{V}r_\varepsilon(x) = 0.$$

We are now in position to give a proof of Theorem 3.3.

Proof (**Proof of Theorem 3.3**). Let $\gamma(t), 0 \leq t \leq r_\varepsilon(x)$, be the unique length parametrized g_ε-geodesic between x_0 and x. We denote

$$\phi(t) = Lr_\varepsilon(\gamma(t)), \quad 0 < t \leq r_\varepsilon(x).$$

From Theorem 3.2, we get the differential inequality

$$-\phi'(t) \geq \frac{1}{n}(\phi(t))^2 + \left(\rho_1(t) - \frac{\kappa(t)}{\varepsilon}\right)\Gamma(r_\varepsilon)(x) + \rho_2(t)\Gamma^\mathcal{V}(r_\varepsilon)(x), \qquad (3.2)$$

because $\Gamma(r_\varepsilon)$ and $\Gamma^\mathcal{V}(r_\varepsilon)$ are constants along γ. We now notice the lower bound

$$\frac{1}{n}(\phi(t))^2 \geq 2\frac{G'(t)}{G(t)}\phi(t) - n\frac{G'(t)^2}{G(t)^2}.$$

Using this lower bound in (3.2), multiplying by $G(t)^2$, and integrating from 0 to $r_\varepsilon(x)$ yields the expected result thanks to lemma 3.5. $\qquad\square$

The proof of Theorem 3.4 is identical.

Corollary 3.6. *Assume that the functions ρ_1, κ, ρ_2 are constant. Denote*

$$\kappa_\varepsilon = \min\left(\rho_1 - \frac{\kappa}{\varepsilon}, \frac{\rho_2}{\varepsilon}\right).$$

For $x \neq x_0 \in \mathbb{M}$, not in the d_ε cut-locus of x_0

$$\Delta_\mathcal{H} r_\varepsilon(x) \leq \begin{cases} \sqrt{n\kappa_\varepsilon}\cot(\sqrt{\frac{\kappa_\varepsilon}{n}}r_\varepsilon(x)), & \text{if } \kappa_\varepsilon > 0, \\ \dfrac{n}{r_\varepsilon(x)}, & \text{if } \kappa_\varepsilon = 0, \\ \sqrt{n|\kappa_\varepsilon|}\coth(\sqrt{\frac{|\kappa_\varepsilon|}{n}}r_\varepsilon(x)), & \text{if } \kappa_\varepsilon < 0. \end{cases} \qquad (3.3)$$

Corollary 3.7. *Assume that the function ρ_3 is constant. Then, for $x \neq x_0$ not in the cut-locus of x_0,*

$$\Delta_\mathcal{V} r_\varepsilon(x) \leq F(r_\varepsilon(x), \Gamma^\mathcal{V}(r_\varepsilon)(x))$$

where

$$F(r_\varepsilon, \Gamma^\mathcal{V}(r_\varepsilon)) = \begin{cases} \sqrt{m\rho_3\Gamma^\mathcal{V}(r_\varepsilon)}\cot\left(\sqrt{\frac{\rho_3\varepsilon^2\Gamma^\mathcal{V}(r_\varepsilon)}{m}}r_\varepsilon\right), & \text{if } \rho_3 > 0, \\ \dfrac{m}{\varepsilon r_\varepsilon}, & \text{if } \rho_3 = 0, \\ \sqrt{-m\rho_3\Gamma^\mathcal{V}(r_\varepsilon)}\coth\left(\sqrt{-\frac{\rho_3\varepsilon^2\Gamma^\mathcal{V}(r_\varepsilon)}{m}}r_\varepsilon\right), & \text{if } \rho_3 < 0. \end{cases}$$

3.4 Li-Yau inequality and volume doubling properties for the sub-Riemannian distance

Throughout the section we assume that globally on $\mathbb{M}$, for every $X \in \Gamma^\infty(\mathcal{H})$ and $Z \in \Gamma^\infty(\mathcal{V})$,

$$\mathbf{Ric}_\mathcal{H}(X, X) \geq \rho_1\|X\|_\mathcal{H}^2, \quad -\langle \mathbf{J}^2 X, X\rangle_\mathcal{H} \leq \kappa\|X\|_\mathcal{H}^2, \quad -\frac{1}{4}\mathbf{Tr}_\mathcal{H}(J_Z^2) \geq \rho_2\|Z\|_\mathcal{V}^2,$$

for some $\rho_1 \in \mathbb{R}$, $\kappa, \rho_2 > 0$. We also assume that the horizontal distribution $\mathcal{H}$ is Yang-Mills, which means that

$$\delta_{\mathcal{H}} T = 0.$$

We show in this section how to obtain the Li-Yau estimate.

Henceforth, we will indicate $C_b^{\infty}(\mathbb{M}) = C^{\infty}(\mathbb{M}) \cap L^{\infty}(\mathbb{M})$ and by P_t the semigroup generated by $\Delta_{\mathcal{H}}$. A key lemma is the following.

Lemma 3.8. *Let $f \in C_b^{\infty}(\mathbb{M})$, $f > 0$ and $T > 0$, and consider the functions*

$$\phi_1(x,t) = (P_{T-t}f)(x)\Gamma(\ln P_{T-t}f)(x),$$

$$\phi_2(x,t) = (P_{T-t}f)(x)\Gamma^{\mathcal{V}}(\ln P_{T-t}f)(x),$$

which are defined on $\mathbb{M} \times [0,T)$. We have

$$\Delta_{\mathcal{H}}\phi_1 + \frac{\partial \phi_1}{\partial t} = 2(P_{T-t}f)\Gamma_2(\ln P_{T-t}f).$$

and

$$\Delta_{\mathcal{H}}\phi_2 + \frac{\partial \phi_2}{\partial t} = 2(P_{T-t}f)\Gamma_2^{\mathcal{V}}(\ln P_{T-t}f).$$

Proof. This is direct computation without trick. Let us just point out that the formula

$$\Delta_{\mathcal{H}}\phi_2 + \frac{\partial \phi_2}{\partial t} = 2(P_{T-t}f)\Gamma_2^{\mathcal{V}}(\ln P_{T-t}f).$$

uses the fact that $\Gamma(g, \Gamma^{\mathcal{V}}(g)) = \Gamma^{\mathcal{V}}(g, \Gamma(g))$ and thus that the foliation is totally geodesic. $\qquad \square$

We will need the following classical lemma.

Lemma 3.9. *Let $T > 0$. Let $u, v : \mathbb{M} \times [0,T] \to \mathbb{R}$ be smooth functions such that for every $T > 0$, $\sup_{t \in [0,T]} \|u(\cdot, t)\|_{\infty} < \infty$, $\sup_{t \in [0,T]} \|v(\cdot, t)\|_{\infty} < \infty$; If the inequality*

$$\Delta_{\mathcal{H}}u + \frac{\partial u}{\partial t} \geq v$$

holds on $\mathbb{M} \times [0,T]$, then we have

$$P_T(u(\cdot, T))(x) \geq u(x, 0) + \int_0^T P_s(v(\cdot, s))(x) ds.$$

We now show how to prove the Li-Yau estimates for the horizontal semigroup P_t.

Theorem 3.10. *Let $\alpha > 2$. For $f \in C_0^{\infty}(\mathbb{M})$, $f \geq 0$, $f \neq 0$, the following inequality holds for $t > 0$:*

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{\alpha} t \Gamma^{\mathcal{V}}(\ln P_t f)$$

$$\leq \left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2} - \frac{2\rho_1}{\alpha}t\right)\frac{\Delta_{\mathcal{H}}P_t f}{P_t f} + \frac{n\rho_1^2}{2\alpha}t - \frac{\rho_1 n}{2}\left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2}\right) + \frac{n(\alpha-1)^2\left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2}\right)^2}{8(\alpha-2)t}.$$

23

Proof. We fix $T > 0$ and consider two functions $a, b : [0, T] \to \mathbb{R}_{\geq 0}$ to be chosen later. Let $f \in C^{\infty}(\mathbb{M})$, $f \geq 0$. Consider the function

$$\phi(x, t) = a(t)(P_{T-t}f)(x)\Gamma(\ln P_{T-t}f)(x) + b(t)(P_{T-t}f)(x)\Gamma^{\mathcal{V}}(\ln P_{T-t}f)(x).$$

Applying Lemma 3.8 and the curvature-dimension inequality in Theorem 3.2, we obtain

$$\begin{aligned}
&\Delta_{\mathcal{H}}\phi + \frac{\partial \phi}{\partial t} \\
&= a'(P_{T-t}f)\Gamma(\ln P_{T-t}f) + b'(P_{T-t}f)\Gamma^{\mathcal{V}}(\ln P_{T-t}f) + 2a(P_{T-t}f)\Gamma_2(\ln P_{T-t}f) \\
&\quad + 2b(P_{T-t}f)\Gamma_2^{\mathcal{V}}(\ln P_{T-t}f) \\
&\geq \left(a' + 2\rho_1 a - 2\kappa\frac{a^2}{b}\right)(P_{T-t}f)\Gamma(\ln P_{T-t}f) + (b' + 2\rho_2 a)(P_{T-t}f)\Gamma^{\mathcal{V}}(\ln P_{T-t}f) \\
&\quad + \frac{2a}{n}(P_{T-t}f)(\Delta_{\mathcal{H}}(\ln P_{T-t}f))^2.
\end{aligned}$$

But, for any function $\gamma : [0, T] \to \mathbb{R}$

$$(\Delta_{\mathcal{H}}(\ln P_{T-t}f))^2 \geq 2\gamma\Delta_{\mathcal{H}}(\ln P_{T-t}f) - \gamma^2,$$

and from chain rule

$$\Delta_{\mathcal{H}}(\ln P_{T-t}f) = \frac{\Delta_{\mathcal{H}}P_{T-t}f}{P_{T-t}f} - \Gamma(\ln P_{T-t}f).$$

Therefore, we obtain

$$\begin{aligned}
\Delta_{\mathcal{H}}\phi + \frac{\partial \phi}{\partial t} &\geq \left(a' + 2\rho_1 a - 2\kappa\frac{a^2}{b} - \frac{4a\gamma}{n}\right)(P_{T-t}f)\Gamma(\ln P_{T-t}f) \\
&\quad + (b' + 2\rho_2 a)(P_{T-t}f)\Gamma^{\mathcal{V}}(\ln P_{T-t}f) + \frac{4a\gamma}{n}\Delta_{\mathcal{H}}P_{T-t}f - \frac{2a\gamma^2}{n}P_{T-t}f.
\end{aligned}$$

The idea is now to chose a, b, γ such that

$$\begin{cases} a' + 2\rho_1 a - 2\kappa\frac{a^2}{b} - \frac{4a\gamma}{n} = 0 \\ b' + 2\rho_2 a = 0 \end{cases}$$

With this choice we get

$$\Delta_{\mathcal{H}}\phi + \frac{\partial \phi}{\partial t} \geq \frac{4a\gamma}{n}\Delta_{\mathcal{H}}P_{T-t}f - \frac{2a\gamma^2}{n}P_{T-t}f \tag{3.4}$$

We wish to apply Lemma 3.9. We take now $f \in C_0^{\infty}(\mathbb{M})$ and apply the previous inequality with $f_{\varepsilon} = f + \varepsilon$ instead of f, where $\varepsilon > 0$. If moreover $a(T) = b(T) = 0$, we end up with the inequality

$$\begin{aligned}
&a(0)(P_T f_{\varepsilon})(x)\Gamma(\ln P_T f_{\varepsilon})(x) + b(0)(P_T f)(x)\Gamma^{\mathcal{V}}(\ln P_T f_{\varepsilon})(x) \\
&\leq -\int_0^T \frac{4a\gamma}{n}dt\Delta_{\mathcal{H}}P_T f_{\varepsilon}(x) + \int_0^T \frac{2a\gamma^2}{n}dt P_T f_{\varepsilon}(x)
\end{aligned} \tag{3.5}$$

If we now chose $b(t) = (T-t)^\alpha$ and b, γ such that

$$\begin{cases} a' + 2\rho_1 a - 2\kappa \frac{a^2}{b} - \frac{4a\gamma}{n} = 0 \\ b' + 2\rho_2 a = 0 \end{cases}$$

the result follows by a simple computation and sending then $\varepsilon \to 0$. $\qquad\square$

Observe that if $\mathfrak{Ric}_\mathcal{H} \geq 0$, then we can take $\rho_1 = 0$ and the estimate simplifies to

$$\Gamma(\ln P_t f) + \frac{2\rho_2}{\alpha} t \Gamma^\mathcal{V}(\ln P_t f) \leq \left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2}\right) \frac{\Delta_\mathcal{H} P_t f}{P_t f} + \frac{n(\alpha-1)^2 \left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2}\right)^2}{8(\alpha-2)t}.$$

By adapting a classical method of Li and Yau and integrating this last inequality on sub-Riemannian geodesics leads to a parabolic Harnack inequality (details are in [16]). For $\alpha > 2$, we denote

$$D_\alpha = \frac{n(\alpha-1)^2 \left(1 + \frac{\alpha\kappa}{(\alpha-1)\rho_2}\right)}{4(\alpha-2)}. \tag{3.6}$$

The minimal value of D_α is difficult to compute, depends on κ, ρ_2 and does not seem relevant because the constants we get are anyhow not optimal. We just point out that the choice $\alpha = 3$ turns out to simplify many computations and is actually optimal when $\kappa = 4\rho_2$.

Corollary 3.11. *Let us assume that $\rho_1 \geq 0$. Let $f \in L^\infty(\mathbb{M})$, $f \geq 0$, and consider $u(x,t) = P_t f(x)$. For every $(x,s), (y,t) \in \mathbb{M} \times (0,\infty)$ with $s < t$ one has with D_α as in (3.6)*

$$u(x,s) \leq u(y,t) \left(\frac{t}{s}\right)^{\frac{D_\alpha}{2}} \exp\left(\frac{D_\alpha}{n} \frac{d(x,y)^2}{4(t-s)}\right).$$

Here $d(x,y)$ is the sub-Riemannian distance between x and y.

It is classical since the work by Li and Yau and not difficult to prove that a parabolic Harnack inequality implies a Gaussian upper bound on the heat kernel. With the curvature dimension inequality in hand, it is actually also possible, but much more difficult, to prove a lower bound. The final result proved in [15] is:

Theorem 3.12. *Let us assume that $\rho_1 \geq 0$, then for any $0 < \varepsilon < 1$ there exists a constant $C(\varepsilon) = C(n, \kappa, \rho_2, \varepsilon) > 0$, which tends to ∞ as $\varepsilon \to 0^+$, such that for every $x, y \in \mathbb{M}$ and $t > 0$ one has*

$$\frac{C(\varepsilon)^{-1}}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{D_\alpha d(x,y)^2}{n(4-\varepsilon)t}\right) \leq p_t(x,y) \leq \frac{C(\varepsilon)}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right),$$

where $p_t(x,y)$ is the heat kernel of $\Delta_\mathcal{H}$.

This theorem implies the following important result:

Theorem 3.13. *Let us assume that $\rho_1 \geq 0$. Then the metric measure space $(\mathbb{M}, d, \mu)$ satisfies the global volume doubling property and supports a 2-Poincaré inequality.*

4 Sobolev and isoperimetric inequalities

In this section, we study Sobolev and isoperimetric inequalities for the sub-Laplacian. We work in the framework of Section 3 and keep the same notations. Throughout the section we assume that globally on $\mathbb{M}$, for every $X \in \Gamma^\infty(\mathcal{H})$ and $Z \in \Gamma^\infty(\mathcal{V})$,

$$\mathbf{Ric}_{\mathcal{H}}(X, X) \geq \rho_1 \|X\|_{\mathcal{H}}^2, \quad -\langle \mathbf{J}^2 X, X \rangle_{\mathcal{H}} \leq \kappa \|X\|_{\mathcal{H}}^2, \quad -\frac{1}{4}\mathbf{Tr}_{\mathcal{H}}(J_Z^2) \geq \rho_2 \|Z\|_{\mathcal{V}}^2,$$

for some $\rho_1 \in \mathbb{R}$, $\kappa, \rho_2 > 0$.
As seen before, it implies that for every $f \in C^\infty(\mathbb{M})$ one has

$$\Gamma_2^{\mathcal{H}}(f, f) + \varepsilon \Gamma_2^{\mathcal{H}, \mathcal{V}}(f, f) \geq \frac{1}{n}(\Delta_{\mathcal{H}} f)^2 + \left(\rho_1 - \frac{\kappa}{\varepsilon}\right) \Gamma(f, f) + \rho_2 \Gamma^{\mathcal{V}}(f, f).$$

In the sub-Riemannian structure $(\mathbb{M}, \mathcal{H}, g_{\mathcal{H}})$, a notion of intrinsic horizontal perimeter associated to can be defined as follows .

4.1 BV functions in sub-Riemannian manifolds

Let us first observe that, given any point $x \in \mathbb{M}$ there exists an open set $x \in U \subset \mathbb{M}$ in which the operator $\Delta_{\mathcal{H}}$ can be written as

$$\Delta_{\mathcal{H}} = -\sum_{i=1}^{m} X_i^* X_i, \tag{4.1}$$

where the vector fields X_i have Lipschitz continuous coefficients in U, and X_i^* indicates the formal adjoint of X_i in $L^2(\mathbb{M}, d\mu)$.
We indicate with $\mathcal{F}(\mathbb{M})$ the set of C^1 vector fields which are subunit for $\Delta_{\mathcal{H}}$. Given a function $f \in L^1_{loc}(\mathbb{M})$, which is supported in U we define the horizontal total variation of f as

$$\mathrm{Var}(f) = \sup_{\phi \in \mathcal{F}(\mathbb{M})} \int_U f \left(\sum_{i=1}^{m} X_i^* \phi_i\right) d\mu,$$

where on U, $\phi = \sum_{i=1}^{m} \phi_i X_i$. For functions not supported in U, $\mathrm{Var}(f)$ may be defined by using a partition of unity. The space

$$BV(\mathbb{M}) = \{f \in L^1(\mathbb{M}) \mid \mathrm{Var}(f) < \infty\},$$

endowed with the norm
$$\|f\|_{BV(\mathbb{M})} = \|f\|_{L^1(\mathbb{M})} + \mathrm{Var}(f),$$

is a Banach space. As in the Riemannian case, $W^{1,1}(\mathbb{M}) = \{f \in L^1(\mathbb{M}) \mid \sqrt{\Gamma f} \in L^1(\mathbb{M})\}$ is a strict subspace of $BV(\mathbb{M})$ and when $f \in W^{1,1}(\mathbb{M})$ one has in fact

$$\mathrm{Var}(f) = \|\sqrt{\Gamma(f)}\|_{L^1(\mathbb{M})}.$$

Given a measurable set $E \subset \mathbb{M}$ we say that it has finite perimeter, or is a Cacciopoli set if $\mathbf{1}_E \in BV(\mathbb{M})$. In such case the horizontal perimeter of E is by definition

$$P(E) = \mathrm{Var}(\mathbf{1}_E).$$

As in the Riemannian case, we have the following approximation result, see Theorem 1.14 in [25].

Lemma 4.1. *Let $f \in BV(\mathbb{M})$, then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $C_0^\infty(\mathbb{M})$ such that:*

(i) $\|f_n - f\|_{L^1(\mathbb{M})} \to 0$;

(ii) $\int_{\mathbb{M}} \sqrt{\Gamma(f_n)}d\mu \to \mathrm{Var}(f)$.

4.2 Pseudo-Poincaré inequalities

For $p \geq 1$, we define the Sobolev space $W^{1,p}(\mathbb{M})$ as the closure of $C_0^\infty(\mathbb{M})$ with respect to the norm $\|f\|_p + \|\sqrt{\Gamma(f)}\|_p$. The following inequality can be proved from the generalized curvature dimension inequality with $\rho_1 \geq 0$.

Proposition 4.2. *Let $1 \leq p < +\infty$. There exists a constant $C_p > 0$ such that for every $f \in W^{1,p}(\mathbb{M})$ we have*

$$\|f - P_t f\|_p \leq C_p \sqrt{t}\|\sqrt{\Gamma(f)}\|_p.$$

4.3 Sobolev embeddings

For $\alpha < 0$, we define the Besov norm $\|\cdot\|_{B^\alpha_{\infty,\infty}}$ on $\mathbb{M}$ as follows :

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{t>0} t^{-\alpha/2}\|P_t f\|_\infty. \tag{4.2}$$

It is clear from this definition that $\|f\|_{B^\alpha_{\infty,\infty}} \leq 1$ is equivalent to the fact that for every $u > 0$, $|P_{t_u} f| \leq u$ where $t_u = u^{2/\alpha}$.

Theorem 4.3 (Improved Sobolev embedding). *For every $1 \leq p < q < \infty$ and every $f \in W^{1,p}(\mathbb{M})$, we have*

$$\|f\|_q \leq C\|\sqrt{\Gamma(f)}\|_p^\theta \|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}}^{1-\theta} \tag{4.3}$$

where $\theta = \frac{p}{q}$ and where $C > 0$ is a constant that only depends on p, q, ρ_2, κ, d.

Proof. The proof follows [33]; for the sake of completeness, we reproduce the main arguments and make sure they adapt to our sub-Riemannian framework. The proof proceeds in three steps.

Step 1. We first prove the weak-type inequality

$$\|f\|_{q,\infty} \leq C\|\sqrt{\Gamma(f)}\|_p^\theta \|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}}^{1-\theta}.$$

27

Without loss of generality, we can assume $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} \le 1$, which is equivalent to the condition:

$$|P_{t_u}f| \le u \ , \ t_u = u^{2(\theta-1)/\theta} \text{ for every } u > 0. \tag{4.4}$$

We have then

$$u^q \mu\{|f| > 2u\} \ \le \ u^q \mu\{|f - P_{t_u}f| > u\} \ \le \ u^{q-p} \int_M |f - P_{t_u}f|^p d\mu$$

From Proposition 4.2, we have

$$\|f - P_t f\|_p \le C_p \sqrt{t} \|\sqrt{\Gamma(f)}\|_p.$$

Since $q - p + \frac{p}{2}\frac{2(\theta-1)}{\theta} = 0$, we conclude

$$u^q \mu\{|f| > 2u\} \le u^{q-p}\left(C^p t_u^{p/2}\|\sqrt{\Gamma(f)}\|_p^p\right)$$
$$\le C^p \|\sqrt{\Gamma(f)}\|_p^p$$

We finally observe that $\sup_{u>0} u^q \mu\{|f| > 2u\} = \frac{1}{2^q}\|f\|_{q,\infty}^q$, to conclude Step 1.

Step 2. In the previous weak type inequality, we would like to replace the $L^{q,\infty}$-norm by the L^q-norm. Again, we assume $\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} \le 1$, that is $|P_{t_u}f| \le u$ for $t_u = u^{2(\theta-1)/\theta}$, $\forall u > 0$. For $f \in W^{1,p}(\mathbb{M}) \cap L^q(\mathbb{M})$ such that $|P_{t_u}f| \le u$, $\forall u > 0$, we want to show that for some constant $C > 0$,

$$\int_M |f|^q d\mu \le C \int_{\mathbb{M}} \Gamma(f)^{p/2} d\mu.$$

Let $c \ge 5$ be an arbitrary constant. For any $u > 0$, we introduce the truncation

$$\tilde{f}_u = (f - u)^+ \wedge ((c-1)u) + (f + u)^- \vee (-(c-1)u).$$

That is, $\tilde{f}_u(x) = f(x) - u$ when $u \le f(x) \le cu$, and $\tilde{f}_u(x) = f(x) + u$ when $-cu \le f(x) \le -u$, otherwise $|\tilde{f}_u|$ is truncated as constants 0 or $(c-1)u$. Observing

$$\{|f| \ge 5u\} \subset \{|\tilde{f}_u| \ge 4u\},$$

yields

$$\int_0^\infty \mu(\{|f| \ge 5u\})d(u^q) \le \int_0^\infty \mu(\{|\tilde{f}_u| \ge 4u\})d(u^q)$$
$$\le \int_0^\infty \mu(\{|\tilde{f}_u - P_{t_u}f| \ge 3u\})d(u^q) \quad (\text{ since } |P_{t_u}(f)| \le u)$$
$$\le \int_0^\infty \mu(\{|\tilde{f}_u - P_{t_u}\tilde{f}_u| \ge u\})d(u^q) + \int_0^\infty \mu(\{P_{t_u}(|f - \tilde{f}_u|) \ge 2u\})d(u^q).$$

28

We now apply the pseudo-Poincaré inequality for $\tilde{f}_u$ as follows,

$$\mu(\{|\tilde{f}_u - P_{t_u}\tilde{f}_u| \geq u\}) \leq u^{-p}\int_{\mathbb{M}}|\tilde{f}_u - P_{t_u}\tilde{f}_u|^p d\mu$$

$$\leq C'u^{-p}t_u^{p/2}\int_{\mathbb{M}}\Gamma(\tilde{f}_u)^{p/2}d\mu$$

$$= C'u^{-q}\int_{\{u\leq|f|\leq cu\}}\Gamma(f)^{p/2}d\mu.$$

So by integration we get,

$$\int_0^\infty \mu(\{|\tilde{f}_u - P_{t_u}\tilde{f}_u| \geq u\})d(u^q) \leq \int_0^\infty C'qu^{-1}\int_{\{u\leq|f|\leq cu\}}\Gamma(f)^{p/2}d\mu du$$

$$\leq C'q\int_{\mathbb{M}}\Gamma(f)^{p/2}\int_{|f|/c}^{|f|}\frac{du}{u}d\mu$$

$$= C'q\ln c\int_{\mathbb{M}}\Gamma(f)^{p/2}d\mu.$$

On the other hand, we have

$$|f - \tilde{f}_u| = |f - \tilde{f}_u|\,1_{\{|f|\leq cu\}} + |f - \tilde{f}_u|\,1_{\{|f|>cu\}}$$
$$= \min(u,|f|)\,1_{\{|f|\leq cu\}} + (|f| - (c-1)u)\,1_{\{|f|>cu\}} \leq u + |f|\,1_{\{|f|>cu\}}.$$

By integrating, we obtain then

$$\int_0^\infty \mu(\{P_{t_u}(|f - \tilde{f}_u|) \geq 2u\})d(u^q) \leq \int_0^\infty \mu(\{P_{t_u}(|f|\,1_{\{|f|>cu\}}) \geq u\})d(u^q)$$

$$\leq \int_0^\infty \frac{1}{u}\left(\int_{\mathbb{M}}(|f|\,1_{\{|f|>cu\}})d\mu\right)d(u^q) \quad (P_t \text{ is a contraction on } L^1(\mathbb{M}))$$

$$= \frac{q}{q-1}\int_{\mathbb{M}}|f|\left(\int_0^\infty 1_{\{|f|>cu\}}d(u^{q-1})\right)d\mu$$

$$= \frac{q}{q-1}\ \frac{1}{c^{q-1}}\|f\|_q^q.$$

Gathering all the estimates, we can then conclude

$$\frac{1}{5^q}\int_{\mathbb{M}}|f|^q d\mu = \frac{1}{5^q}\|f\|_q^q = \int_0^\infty \mu(\{|f| \geq 5u\})d(u^q)$$

$$\leq C'q\ln c\int_{\mathbb{M}}\Gamma(f)^{p/2}d\mu + \frac{q}{q-1}\ \frac{1}{c^{q-1}}\|f\|_q^q$$

If we pick a large $c \geq 5$ depending on q such that $\frac{1}{5^q} > \frac{q}{q-1}\ \frac{1}{c^{q-1}}$, we have proved the claim

$$\|f\|_q^q \leq C^q\|\sqrt{\Gamma(f)}\|^p$$

with $C = \left(\dfrac{C'q \ln c}{\frac{1}{5^q} - \frac{q}{(q-1)c^{q-1}}} \right)^{1/q}$.

Step 3. Finally, it remains to prove $\|f\|_q < \infty$ is actually a consequence of $\|\sqrt{\Gamma(f)}\|_p < \infty$, $\|f\|_{B^{\theta/(\theta-1)}_{\infty,\infty}} \leq 1$, so that we can remove the condition $f \in L^q(\mathbb{M})$ from Step 2 and complete the proof of theorem. From the weak type inequality of Step 1, we have $\|f\|_{q,\infty} < \infty$. For any $0 < \epsilon < 1$, we define

$$N_\epsilon(f) = \int_\epsilon^{1/\epsilon} \mu(\{|f| \geq 5u\}) d(u^q) \leq \frac{2q}{5^q} \left(\ln \frac{1}{\epsilon} \right) \|f\|_{q,\infty}^q < \infty.$$

Following the argument in Step 2 again, we see that

$$N_\epsilon(f) \leq C'q \ln c \int_\mathbb{M} \Gamma(f)^{p/2} d\mu + \int_\epsilon^{1/\epsilon} \frac{1}{u} \left(\int_\mathbb{M} (|f| \, 1_{\{|f|>cu\}}) d\mu \right) d(u^q).$$

The first term is bounded, and the second term can be estimated as follows.

$$\int_\epsilon^{1/\epsilon} \frac{1}{u} \left(\int_\mathbb{M} (|f| \, 1_{\{|f|>cu\}}) d\mu \right) d(u^q)$$

$$= \int_\epsilon^{1/\epsilon} \frac{1}{u} \left(cu\mu(\{|f| > cu\}) + c \int_u^\infty \mu(\{|f| > cv\}) dv \right) d(u^q)$$

$$\leq (c + \frac{c}{q-1}) \int_\epsilon^{1/\epsilon} \mu(\{|f| \geq cu\}) d(u^q) + \frac{cq}{(q-1)\epsilon^{q-1}} \int_{1/\epsilon}^\infty \mu(\{|f| \geq cu\}) du$$

$$\leq \frac{q}{q-1} \frac{5^q}{c^{q-1}} N_\epsilon(f) + \frac{cq}{q-1} \int_{5/c\epsilon}^{1/\epsilon} \frac{\|f\|_{q,\infty}^q}{(cu)^q} d(u^q) + \frac{cq}{(q-1)\epsilon^{q-1}} \int_{1/\epsilon}^\infty \frac{\|f\|_{q,\infty}^q}{(cu)^q} du$$

$$= \frac{q}{q-1} \frac{5^q}{c^{q-1}} N_\epsilon(f) + \frac{q}{q-1} \frac{1}{c^{q-1}} \|f\|_{q,\infty}^q \left(q \ln \frac{c}{5} + \frac{1}{q-1} \right)$$

So, by choosing c large enough, we have $\sup_{0<\epsilon<1} N_\epsilon(f) < \infty$ which implies $\|f\|_q = \lim_{\epsilon \to 0} 5(N_\epsilon(f))^{1/q} < \infty$. This completes the proof.

$\square$

4.4 Isoperimetric inequalities

Theorem 4.4. *Let $D > 1$. Let us assume that $\mathbb{M}$ is not compact in the metric topology, then the following assertions are equivalent:*

(1) There exists a constant $C_1 > 0$ such that for every $x \in \mathbb{M}$, $r \geq 0$,

$$\mu(B(x,r)) \geq C_1 r^D.$$

(2) There exists a constant $C_2 > 0$ such that for $x \in \mathbb{M}$, $t > 0$,

$$p(x,x,t) \leq \frac{C_2}{t^{\frac{D}{2}}}.$$

(3) For some $1 \leq p, q, r < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$, there exists a constant $C_3 > 0$ such that $\forall f \in C_0^\infty(\mathbb{M})$, we have

$$\|f\|_q \leq C_3 \|\sqrt{\Gamma(f)}\|_p^{p/q} \, \|f\|_r^{1-p/q}.$$

(4) There exists a constant $C_4 > 0$ such that for every Caccioppoli set $E \subset \mathbb{M}$ one has

$$\mu(E)^{\frac{D-1}{D}} \leq C_4 P(E).$$

Remark 4.5. *if we replace the condition of (3) by for all $1 \leq p, q, r < \infty$ with $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$, (1),(2), (3) and (4) would still be equivalent.*

Proof. That $(1) \to (2)$ follows immediately from the Li-Yau Gaussian upper bound

$$p(x, x, t) \leq \frac{C}{\mu(B(x, \sqrt{t}))}.$$

The proof that $(2) \to (3)$ follows from the improved Sobolev embedding Theorem 4.3. Indeed, (2) implies first that for $x, y \in \mathbb{M}$,

$$\begin{aligned}
p(x, y, t) &= \int_\mathbb{M} p(x, z, t/2) p(z, y, t/2) \mu(dy) \\
&\leq \sqrt{\int_\mathbb{M} p(x, z, t/2)^2 \mu(dz)} \sqrt{\int_\mathbb{M} p(y, z, t/2)^2 \mu(dz)} \\
&= \sqrt{p(x, x, t) p(y, y, t)} \\
&\leq \frac{C_2}{t^{\frac{D}{2}}}.
\end{aligned}$$

Therefore, for every $f \in L^1(\mathbb{M})$, we have

$$\|P_t(f)\|_\infty = \left\| \int_\mathbb{M} p(\cdot, y, t) f(y) \mu(dy) \right\|_\infty \leq \|p(\cdot, y, t)\|_\infty \|f\|_1 \leq \frac{C_2}{t^{D/2}} \|f\|_1.$$

On the other hand, P_t is a contraction on $L^\infty(\mathbb{M})$, i.e. $\|P_t\|_{\infty \to \infty} \leq 1$. Therefore, by the Riesz-Thorin interpolation theorem, we deduce that we have the following heat semigroup embedding

$$\|P_t\|_{r \to \infty} \leq \frac{C_2^{1/r}}{t^{D/2r}}, \quad r \geq 1.$$

Let now $1 \leq p, q, r < \infty$ such that $\frac{1}{q} = \frac{1}{p} - \frac{r}{qD}$. Since for $\theta = \frac{p}{q}$, $-\frac{\theta}{2(\theta-1)} - \frac{D}{2r} = 0$, we have

$$\begin{aligned}
\|f\|_{B_{\infty,\infty}^{\theta/(\theta-1)}} &= \sup_{t>0} t^{-\theta/2(\theta-1)} \|P_t f\|_\infty \\
&\leq \sup_{t>0} t^{-\theta/2(\theta-1)} \frac{C_2^{1/r}}{t^{D/2r}} \|f\|_r = C_2^{1/r} \|f\|_r,
\end{aligned}$$

we can conclude (3) from the improved Sobolev embeddding of Theorem 4.3.

The proof that (3) is equivalent to (4) follows the classical ideas of Fleming-Rishel and Maz'ya, and it is based on a generalization of Federer's co-area formula for the space $BV(\mathbb{M})$.

Finally, we show that (3) → (1). We adapt an idea in [47] (see Theorem 3.1.5 on p. 58). For any fix $x \in \mathbb{M}$, $s > 0$, consider the function

$$f(y) = \max\{s - d(x, y), 0\}.$$

Then, it is easily seen that

$$\|f\|_q \geq (s/2)\mu(B(x, s/2))^{1/q}$$
$$\|f\|_r \leq s\mu(B(x, s))^{1/r}$$
$$\|\sqrt{\Gamma(f)}\|_p \leq \mu(B(x, s))^{1/p}.$$

Hence, from (3) we have

$$\mu(B(x, s/2))^{1/q} \leq 2C_3 s^{-p/q} \mu(B(x, s))^{1/q + (1/r)(1 - p/q)}$$
$$= 2C_3 s^{-p/q} \mu(B(x, s))^{1/q + p/qD}.$$

This can be written as follows.

$$\mu(B(x, s)) \geq (2C_3)^{-Dq/(D+p)} s^{Dp/(D+p)} \mu(B(x, s/2))^{D/(D+p)}.$$

$$\mu(B(x, s)) \geq \{(2C_3)^{-q} s^p\}^a \mu(B(x, s/2))^a$$

where $a = D/(D + p) < 1$. Replacing s by $s/2$ iteratively, we obtain

$$\mu(B(x, s)) \geq (2C_3)^{-q(\sum_{j=1}^i a^j)} s^{p(\sum_{j=1}^i a^j)} 2^{-p(\sum_{j=1}^i (j-1)a^j)} \mu(B(x, s/2^i))^{a^i}.$$

From the volume doubling property proved in [15], we have the control

$$\mu(B(x, s/2^i)) \geq C^{-1}(1/2^i)^Q \mu(B(x, s)),$$

for some $C = C(\rho_1, \rho_2, \kappa, d) > 0$ and $Q = \log_2 C$.
Therefore, we have

$$\liminf_{i \to \infty} \mu(B(x, s/2^i))^{a^i} \geq \lim_{i \to \infty} (C^{-1}\mu(B(x, s)))^{a^i}(1/2)^{iQa^i} = 1.$$

Since $\sum_{j=1}^\infty a^j = D/p$, $\sum_{j=1}^\infty (j - 1)a^j = D^2/p^2$, we obtain the volume growth control

$$\mu(B(x, s)) \geq 2^{-(q+D)D/p} C_3^{-qD/p} s^D.$$

This establishes (1), thus completing the proof.

$\square$

References

[1] A. Agrachev, *Any sub-Riemannian metric has points of smoothness*. Dokl. Math. 79 (2009), no. 1, 45–47.

[2] A. Agrachev, P. Lee, *Bishop and Laplacian comparison theorems on three-dimensional contact sub-Riemannian manifolds with symmetry*. J. Geom. Anal. 25 (2015), no. 1, 512–535.

[3] A. Agrachev, P. Lee, *Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds*. Math. Ann. 360 (2014), no. 1-2, 209–253.

[4] A.Agrachev, D. Barilari, L. Rizzi, *Curvature: a variational approach*. Memoirs of the AMS (in press), 2015.

[5] A. Agrachev, D. Barilari, L. Rizzi, *Sub-Riemannian curvature in contact geometry*. J. Geom. Anal. 27 (2017), no. 1, 366–408

[6] A. Agrachev, I. Zelenko, *Geometry of Jacobi curves. I*. J. Dynam. Control Systems, 8 (2002), no. 1, 93–140.

[7] A. Agrachev, I. Zelenko, *Geometry of Jacobi curves. II*. J. Dynam. Control Systems, (2002), no. 2, 167–215.

[8] D. Barilari, L. Rizzi, *Sharp measure contraction property for generalized H-type Carnot groups*, arXiv:1510.05960 (2017), 17 pp.

[9] D. Barilari, L. Rizzi, *Sub-Riemannian interpolation inequalities: Ideal structures*, arXiv:1705.05380 (2017), 45 pp.

[10] E. Barletta, *On the pseudohermitian sectional curvature of a strictly pseudoconvex CR manifold*. Differential Geom. Appl. 25 (2007), no. 6, 612-631.

[11] E. Barletta, S. Dragomir, *Jacobi fields of the Tanaka-Webster connection on Sasakian manifolds*. Kodai Math. J. 29 (2006), no. 3, 406–454.

[12] F. Baudoin, *Sub-Laplacians and hypoelliptic operators on totally geodesic Riemannian foliations*. Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. 1, 259–321, EMS Ser. Lect. Math., Eur. Math. Soc., Zürich, 2016.

[13] F. Baudoin, *Stochastic analysis on sub-Riemannian manifolds with transverse symmetries*. Ann. Probab. 45 (2017), no. 1, 56–81.

[14] F. Baudoin, M. Bonnefont, *Curvature-dimension estimates for the Laplace-Beltrami operator of a totally geodesic foliation*. 2014. Nonlinear Anal. 126 (2015), 159–169.

[15] F. Baudoin, M. Bonnefont, N. Garofalo, *A sub-Riemannian curvature-dimension inequality, volume doubling property and the Poincare inequality*. Math. Ann. 358 (2014), 3-4, 833–860.

[16] F. Baudoin, N. Garofalo, *Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries.* J. Eur. Math. Soc. (JEMS) 19 (2017), no. 1, 151–219.

[17] F. Baudoin, E. Grong, K. Kuwada, A. Thalmaier, *Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations*, 2017, Arxiv preprint, https://arxiv.org/abs/1706.08489

[18] F. Baudoin, B. Kim, *The Lichnerowicz-Obata theorem on sub-Riemannian manifolds with transverse symmetries.* J. Geom. Anal. 26 (2016), no. 1, 156–170.

[19] F. Baudoin, B. Kim, J. Wang, *Transverse Weitzenböck formulas and curvature dimension inequalities on Riemannian foliations with totally geodesic leaves.* Comm. Anal. Geom. 24 (2016), no. 5, 913–937

[20] A. Besse, *Einstein manifolds.* Reprint of the 1987 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.

[21] C. P. Boyer, K. Galicki, *3-Sasakian manifolds.* Surveys Diff. Geom. 7 (1999) 123–184.

[22] I. Chavel, *Riemannian geometry: a modern introduction.* Cambridge tracts in mathematics, 108, Cambridge University Press, Cambridge, 1993.

[23] S. Dragomir, G. Tomassini, *Differential geometry and analysis on CR manifolds.* Birkhäuser, Vol. 246, 2006.

[24] K.D. Elworthy, Y. Le Jan, X.-M. Li, *On the geometry of diffusion operators and stochastic flows.* Lecture Notes in Mathematics, 1720. Springer-Verlag, Berlin, 1999.

[25] N. Garofalo, D.M. Nhieu: Lipschitz continuity, global smooth approximations and extension theorems for Sobolev functions in Carnot-Carathéodory spaces, J. Anal. Math., **74**, 67-97 (1998)

[26] E. Grong, A. Thalmaier, *Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations: Part I.* Math. Z. 282 (2016), no. 1-2, 99–130.

[27] E. Grong, A. Thalmaier, *Curvature-dimension inequalities on sub-Riemannian manifolds obtained from Riemannian foliations: Part II.* Math. Z. 282 (2016), no. 1-2, 131–164.

[28] E. Grong, A. Thalmaier, *Stochastic completeness and gradient representations for sub-Riemannian manifolds*, arXiv:1605.00785 (2016), 35 pp.

[29] G. Hernandez, *On hyper f-structures*, Mathematische Annalen, Volume: 306, Issue: 2, page 205-230, (1996).

[30] N. Hitchin, *On the curvature of rational surfaces.* Proceedings of Symposia in Pure Mathematics, vol. 27, 1975.

[31] A. Kaplan, *Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms.* Trans. Amer. Math. Soc. 258 (1980), no. 1, 147–153.

[32] W. Jelonek, *Positive and negative $3K$-contact structures.* Proceedings of the AMS 129 (2001), no. 1, 247-256.

[33] M. Ledoux, *On improved Sobolev embedding theorems*, Math. Res. Lett. **10** (2003), no. 5-6, 659–669.

[34] P. Lee, *Ricci curvature lower bounds on Sasakian manifolds*, arXiv: 1511.09381 (2015), 30 pp.

[35] P. Lee, *On measure contraction property without Ricci curvature lower bound.* Potential Anal. 44 (2016), no. 1, 27–41.

[36] P. Lee, C. Li, *Bishop and Laplacian comparison theorems on Sasakian manifolds*, arXiv: 1310.5322 (2013), 38 pp.

[37] P. Lee, C. Li, I. Zelenko, *Ricci curvature type lower bounds for sub-Riemannian structures on Sasakian manifolds.* Discrete Contin. Dyn. Syst. 36 (2016), no. 1, 303–321.

[38] C. Li, I. Zelenko, *Differential geometry of curves in Lagrange Grassmannians with given Young diagram.* Differential Geom. Appl. 27 (2009), no. 6, 723–742.

[39] A. Moroianu, U. Semmelmann, *Clifford structures on Riemannian manifolds.* Advances in Mathematics 228 (2011) 940–967

[40] S. Ohta, *On the measure contraction property of metric measure spaces.* Comment. Math. Helv. 82 (2007), no. 4, 805–828.

[41] P. Petersen, *Riemannian geometry.* Graduate Texts in Mathematics, vol. 171, Springer-Verlag, 2006.

[42] L. Rifford, *Ricci curvatures in Carnot groups.* Math. Control Relat. Fields 3 (2013), no. 4, 467–487.

[43] L. Rifford, E. Trélat, *Morse-Sard type results in sub-Riemannian geometry.* Math. Ann. 332 (2005), no. 1, 145–159.

[44] L. Rizzi, *Measure contraction properties of Carnot groups.* Calc. Var. Partial Differential Equations 55 (2016), no. 3, paper no. 60, 20 pp.

[45] L. Rizzi, P. Silveira, *Sub-Riemannian Ricci curvatures and universal diameter bounds for 3-Sasakian manifolds*, arXiv:1509.05410 (2015), 34 pp.

[46] S. Salamon, *Quaternionic Kähler Manifolds.* Invent. Math. 67 (1982), 143–171.

[47] Saloff-Coste, L. *Aspect of Sobolev-type inequalities* London Mathematical Society Lecture Note Series, 289. Cambridge University Press, Cambridge, 2002.

[48] B. Simon, *Convexity. An analytic viewpoint.* Cambridge Tracts in Mathematics, vol. 187. Cambridge University Press, Cambridge, 2011.

[49] K.-T. Sturm, *On the geometry of metric measure spaces I.* Acta Math. 196 (2006), no. 1, 65–131.

[50] K.-T. Sturm, *On the geometry of metric measure spaces II.* Acta Math. 196 (2006), no. 1, 133–177.

[51] S. Tanno, *Variation problems on contact Riemannian manifolds.* Trans. Amer. Math. Soc. 314 (1989), no. 1, 349–379.

[52] S. Tanno, *Remark on a triple of K-contact structures.* Tohoku Math. Journal 48 (1996), 519–531.

[53] P. Tondeur, *Foliations on Riemannian manifolds.* Universitext. Springer-Verlag, New York, 1988.

[54] J. Wang, *Sub-Riemannian heat kernels on model spaces and curvature dimension inequalities on contact manifolds.* Phd dissertation thesis, Purdue University, 2014.

[55] W. Ziller, *Fatness revisited.* Preprint: https://www.math.upenn.edu/ wziller/papers/Fat-09.pdf