An Introduction to Dirichlet Spaces

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Chapter 1

Semigroups

1.1 Semigroups on Banach spaces: The Hille-Yosida theorem

Let $(B, \|\cdot\|)$ be a Banach space, which for us will later be $L^p(X,\mu)$, where (X,μ) is a measure space.

Definition 1.1.1. *A family of bounded operators $(P_t)_{t\geq 0}$ on B is called a strongly continuous contraction semigroup if:*

- $P_0 = \mathbf{Id}$ *and for* $s, t \geq 0$, $P_{s+t} = P_s P_t$;
- *For each* $x \in B$, *the map* $t \to P_t x$ *is continuous;*
- *For each* $x \in B$ *and* $t \geq 0$, $\|P_t x\| \leq \|x\|$.

Now, let us recall that a densely defined linear operator

$$A : \mathcal{D}(A) \subset B \to B$$

is said to be closed if for every sequence $x_n \in \mathcal{D}(A)$ that converges to $x \in B$ and such that $Ax_n \to y \in B$, we have $x \in \mathcal{D}(A)$ and $y = Ax$.

Proposition 1.1.2. *Let $(P_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on B. There exists a closed and densely defined operator*

$$A : \mathcal{D}(A) \subset B \to B$$

where

$$\mathcal{D}(A) = \left\{ f \in B, \quad \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists} \right\},$$

such that for $f \in \mathcal{D}(A)$,

$$\lim_{t\to 0} \left\| \frac{P_t f - f}{t} - Af \right\| = 0.$$

The operator A is called the generator of the semigroup $(P_t)_{t\geq 0}$. We also say that A generates $(P_t)_{t\geq 0}$.

Proof. Let us consider the following bounded operators on B :

$$A_t = \frac{1}{t}\int_0^t P_s ds.$$

For $f \in B$ and $h > 0$, we have

$$\begin{aligned}\frac{1}{t}\left(P_t A_h f - A_h f\right) &= \frac{1}{ht}\int_0^h (P_{s+t}f - P_s f)ds \\ &= \frac{1}{ht}\int_0^t (P_{s+h}f - P_s f)ds.\end{aligned}$$

Therefore, we obtain

$$\lim_{t\to 0}\frac{1}{t}\left(P_t A_h f - A_h f\right) = \frac{1}{h}\left(P_h f - f\right).$$

This implies,

$$\{A_h f, f \in B, h > 0\} \subset \left\{f \in B, \quad \lim_{t\to 0}\frac{P_t f - f}{t} \text{ exists}\right\}$$

Since $\lim_{h\to 0} A_h f = f$, we deduce that

$$\left\{f \in B, \quad \lim_{t\to 0}\frac{P_t f - f}{t} \text{ exists}\right\}$$

is dense in B. We can then consider

$$Af = \lim_{t\to 0}\frac{P_t f - f}{t},$$

which is of course defined on the domain

$$\mathcal{D}(A) = \left\{f \in B, \quad \lim_{t\to 0}\frac{P_t f - f}{t} \text{ exists}\right\}.$$

We let as an exercise to the reader to prove that A is closed (Hint: It is the limit of the bounded operators $\frac{P_t f - f}{t}, t > 0$). □

The following important theorem is due to Hille and Yosida and provides, through spectral properties, a characterization of closed operators that are generators of contraction semigroups.

Let $A : \mathcal{D}(A) \subset B \to B$ be a densely defined closed operator. A constant $\lambda \in \mathbb{R}$ is said to be in the spectrum of A if the operator $\lambda \mathbf{Id} - A$ is not bijective. In that case, it is a consequence of the closed graph theorem[1] that if λ is not in the spectrum of A , then the operator $\lambda \mathbf{Id} - A$ has a bounded inverse. The spectrum of an operator A shall be denoted $\rho(A)$.

[1] An everywhere defined operator between two Banach spaces $A : B_1 \to B_2$ is bounded if and only if it is closed.

Theorem 1.1.3 (Hille-Yosida theorem)**.** *A necessary and sufficient condition that a densely defined closed operator A generates a strongly continuous contraction semigroup is that:*

- $\rho(A) \subset (-\infty, 0]$*;*
- $\|(\lambda \mathbf{Id} - A)^{-1}\| \le \frac{1}{\lambda}$ *for all* $\lambda > 0$*.*

Proof. Let us first assume that A generates a strongly continuous contraction semigroup $(P_t)_{t\ge 0}$. Let $\lambda > 0$. We want to prove that $\lambda \mathbf{Id} - A$ is a bijective operator $\mathcal{D}(A) \to B$. The formal Laplace transform formula

$$\int_0^{+\infty} e^{-\lambda t} e^{tA} dt = (\lambda \mathbf{Id} - A)^{-1},$$

suggests that the operator

$$\mathbf{R}_\lambda = \int_0^{+\infty} e^{-\lambda t} P_t dt$$

is the inverse of $\lambda \mathbf{Id} - A$. We show this is indeed the case. First, let us observe that $\mathbf{R}_\lambda$ is well-defined as a Riemann integral since $t \to P_t$ is continuous and $\|P_t\| \le 1$. We now show that for $x \in B$, $\mathbf{R}_\lambda x \in \mathcal{D}(A)$. For $h > 0$,

$$\begin{aligned}
\frac{P_h - \mathbf{Id}}{h} \mathbf{R}_\lambda x &= \int_0^{+\infty} e^{-\lambda t} \frac{P_h - \mathbf{Id}}{h} P_t x dt \\
&= \int_0^{+\infty} e^{-\lambda t} \frac{P_{h+t} - P_t}{h} x dt \\
&= e^{\lambda h} \int_h^{+\infty} e^{-\lambda s} \frac{P_s - P_{s-h}}{h} x ds \\
&= \frac{e^{\lambda h}}{h} \left(\mathbf{R}_\lambda x - \int_0^h e^{-\lambda s} P_s x ds - \int_h^{+\infty} e^{-\lambda s} P_{s-h} x ds \right) \\
&= \frac{e^{\lambda h} - 1}{h} \mathbf{R}_\lambda x - \frac{e^{\lambda h}}{h} \int_0^h e^{-\lambda s} P_s x ds
\end{aligned}$$

By letting $h \to 0$, we deduce that $\mathbf{R}_\lambda x \in \mathcal{D}(A)$ and moreover

$$A \mathbf{R}_\lambda x = \lambda \mathbf{R}_\lambda x - x.$$

Therefore we proved

$$(\lambda \mathbf{Id} - A) \mathbf{R}_\lambda = \mathbf{Id}.$$

Furthermore, it is readily checked that, since A is closed, for $x \in \mathcal{D}(A)$,

$$A \mathbf{R}_\lambda x = A \int_0^{+\infty} e^{-\lambda t} P_t x dt = \int_0^{+\infty} e^{-\lambda t} A P_t x dt = \int_0^{+\infty} e^{-\lambda t} P_t A x dt = \mathbf{R}_\lambda A x.$$

We therefore conclude

$$(\lambda \mathbf{Id} - A)\mathbf{R}_\lambda = \mathbf{R}_\lambda(\lambda \mathbf{Id} - A) = \mathbf{Id}.$$

Thus,

$$\mathbf{R}_\lambda = (\lambda \mathbf{Id} - A)^{-1},$$

and, from the formula

$$\mathbf{R}_\lambda = \int_0^{+\infty} e^{-\lambda t} P_t dt,$$

it is clear that

$$\|\mathbf{R}_\lambda\| \le \frac{1}{\lambda}.$$

Let us now assume that A is a densely defined closed operator such that

- $\rho(A) \subset (-\infty, 0]$;
- $\|(\lambda \mathbf{Id} - A)^{-1}\| \le \frac{1}{\lambda}$ for all $\lambda > 0$.

The idea is to consider the following sequence of bounded operators

$$A_n = -n\mathbf{Id} + n^2(n\mathbf{Id} - A)^{-1},$$

from which it is easy to define a contraction semigroup and then to show that $A_n \to A$. We will then define a contraction semigroup associated to A as the limit of the contraction semigroups associated to A_n.

First, for $x \in \mathcal{D}(A)$, we have

$$A_n x = n(n\mathbf{Id} - A)^{-1} A x \to_{n\to+\infty} 0.$$

Now, since A_n is a bounded operator, we may define a semigroup $(P_t^n)_{t\ge 0}$ through the formula

$$P_t^n = \sum_{k=0}^{+\infty} \frac{t^k A_n^k}{k!}.$$

At that point, let us observe that we also have

$$P_t^n = e^{-nt} \sum_{k=0}^{+\infty} \frac{n^{2k} t^k (n\mathbf{Id} - A)^{-k}}{k!}.$$

As a consequence, we have

$$\begin{aligned}\|P_t^n\| &\le e^{-nt} \sum_{k=0}^{+\infty} \frac{n^{2k} \|(n\mathbf{Id} - A)^{-1}\|^k}{k!} \\ &\le e^{-nt} \sum_{k=0}^{+\infty} \frac{n^k t^k}{k!} \\ &\le 1\end{aligned}$$

and $(P_t^n)_{t\geq 0}$ is therefore a contraction semigroup. The strong continuity is also easily checked:

$$\begin{aligned}\|P_{t+h}^n - P_t^n\| &= \|P_t^n(P_h^n - \mathbf{Id})\| \\ &\leq \|P_h^n - \mathbf{Id}\| \\ &\leq \sum_{k=1}^{+\infty} \frac{h^k \|A_n\|^k}{k!} \\ &\to_{h\to 0} 0.\end{aligned}$$

We now prove that for fixed $t \geq 0$, $x \in \mathcal{D}(A)$, $(P_t^n x)_{n\geq 1}$ is a Cauchy sequence. We have

$$\begin{aligned}\|P_t^n x - P_t^m x\| &= \left\| \int_0^t \frac{d}{ds}(P_s^n P_{t-s}^m x) ds \right\| \\ &= \left\| \int_0^t P_s^n P_{t-s}^m (A_n x - A_m x) ds \right\| \\ &\leq \int_0^t \|A_n x - A_m x\| ds \\ &\leq t \|A_n x - A_m x\|.\end{aligned}$$

Therefore for $x \in \mathcal{D}(A)$, $(P_t^n x)_{n\geq 1}$ is a Cauchy sequence and we can define

$$P_t x = \lim_{n\to+\infty} P_t^n x.$$

Since $\mathcal{D}(A)$ is dense and the family $(P_t^n)_{n\geq 1}$ uniformly bounded, the above limit actually exists for every $x \in B$, so that $(P_t)_{t\geq 0}$ is well-defined on B. It is clear that $(P_t)_{t\geq 0}$ is a strongly continuous semigroup, inheriting these properties from $(P_t^n)_{t\geq 0}$ (the details are let to the reader here).
It remains to show that the generator of $(P_t)_{t\geq 0}$, call it $\tilde{A}$ is equal to A. For every $t \geq 0$, $x \in \mathcal{D}(A)$ and $n \geq 1$,

$$P_t^n x = x + \int_0^t P_s^n A x ds,$$

therefore

$$P_t^n x = x + \int_0^t P_s^n A x ds.$$

Hence $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$ and for $x \in \mathcal{D}(A)$, $\tilde{A}x = Ax$. Finally, since for $\lambda > 0$, $(\lambda \mathbf{Id} - A)\mathcal{D}(A) = B = (\lambda \mathbf{Id} - \tilde{A})\mathcal{D}(\tilde{A})$, we conclude $\mathcal{D}(A) = \mathcal{D}(\tilde{A})$. □

Exercise 1.1.4. *By using the proof of Theorem 1.1.3, show the following fact: If A_1 and A_2 are the generators of contraction semigroups $(P_t^1)_{t\geq 0}$ and $(P_t^2)_{t\geq 0}$, then for $x \in B$, the two following statements are equivalent:*

- $\forall\ \lambda > 0, \quad (\lambda \mathbf{Id} - A_1)^{-1} x = (\lambda \mathbf{Id} - A_2)^{-1} x;$

- $\forall\, t \geq 0, \quad P_t^1 x = P_t^2 x.$

As powerful as it is, the Hille-Yosida theorem is difficult to directly apply to the theory of diffusion semigroups. The following corollary is then useful.

Definition 1.1.5. *A densely defined operator on a Banach space B is called dissipative if for each $x \in \mathcal{D}(A)$, we can find an element ϕ of the dual space B^*, such that:*

- $\|\phi\| = \|x\|$*;*
- $\phi(x) = \|x\|^2$*;*
- $\phi(Ax) \leq 0$*.*

With this new definition in hands, we have the following corollary of the Hille-Yosida theorem:

Corollary 1.1.6. *A closed operator A on a Banach space B is the generator of a strongly continuous contraction semigroup, if and only if:*

- *A is dissipative;*
- *For $\lambda > 0$, the range of the operator $\lambda \mathbf{Id} - A$ is B.*

Proof. Let us first assume that A is the generator of a contraction semigroup $(P_t)_{t \geq 0}$. From the Hahn-Banach theorem, there exists $\phi \in B^*$ such that $\|\phi\| = \|x\|$ and $\phi(x) = \|x\|^2$. We have, at $t = 0$,

$$\frac{d}{dt}\phi(P_t x) = \phi(Ax),$$

but

$$|\phi(P_t x)| \leq \|\phi\|\|P_t x\| \leq \|\phi\|\|x\| \leq \|x\|^2 \leq \phi(x),$$

thus, at $t = 0$,

$$\frac{d}{dt}\phi(P_t x) \leq 0,$$

and we conclude

$$\phi(Ax) \leq 0.$$

The fact that for $\lambda > 0$, the range of the operator $\lambda \mathbf{Id} - A$ is B is a straightforward consequence of Theorem 1.1.3.
Let us now assume that A is a densely defined closed operator such that:

- A is dissipative;
- For $\lambda > 0$, the range of the operator $\lambda \mathbf{Id} - A$ is B.

Let $x \in \mathcal{D}(A)$ and let $\phi \in B^*$, such that:

- $\|\phi\| = \|x\|$;

- $\phi(x) = \|x\|^2$;
- $\phi(Ax) \le 0$.

For $\lambda > 0$,

$$\begin{aligned}\lambda\|x\|^2 &\le \lambda\phi(x) - \phi(Ax) \\ &\le \phi((\lambda\mathbf{Id} - A)x) \\ &\le \|x\|\|(\lambda\mathbf{Id} - A)x\|.\end{aligned}$$

Thus,

$$\|(\lambda\mathbf{Id} - A)x\| \ge \lambda\|x\|.$$

This implies that the range $\mathcal{R}_\lambda$ of the operator $\lambda\mathbf{Id} - A$ is closed and that this operator has a bounded inverse from $\mathcal{R}_\lambda$ to $\mathcal{D}(A)$ with norm lower than $\frac{1}{\lambda}$. Since $\mathcal{R}_\lambda = B$, the proof is complete. □

1.2 Semigroups on Hilbert spaces: The golden triangle

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ be a Hilbert space and let A be a densely defined operator on a domain $\mathcal{D}(A)$. We have the following basic definitions.

- The operator A is said to be symmetric if for $f, g \in \mathcal{D}(A)$,

$$\langle f, Ag\rangle_{\mathcal{H}} = \langle Af, g\rangle_{\mathcal{H}}.$$

- The operator A is said to be non negative symmetric operator, if it is symmetric and if for $f \in \mathcal{D}(A)$,

$$\langle f, Af\rangle_{\mathcal{H}} \ge 0.$$

  It is said to be non positive, if for $f \in \mathcal{D}(A)$,

$$\langle f, Af\rangle_{\mathcal{H}} \le 0.$$

- The adjoint A^* of A is an operator defined on the domain

$$\mathcal{D}(A^*) = \{f \in \mathcal{H}, \exists\, c(f) \ge 0, \forall\, g \in \mathcal{D}(A), |\langle f, Ag\rangle_{\mathcal{H}}| \le c(f)\|g\|_{\mathcal{H}}\}.$$

  Since for $f \in \mathcal{D}(A^*)$, the map $g \to \langle f, Ag\rangle_{\mathcal{H}}$ is bounded on $\mathcal{D}(A)$, it extends thanks to Hahn-Banach theorem to B. The Riesz representation theorem allows then to define A^* by the formula

$$\langle A^* f, g\rangle_{\mathcal{H}} = \langle f, Ag\rangle_{\mathcal{H}}$$

  where $g \in \mathcal{D}(A), f \in \mathcal{D}(A^*)$. Since $\mathcal{D}(A)$ is dense, A^* is uniquely defined.

- The operator A is said to be self-adjoint if it is symmetric and if $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Let us observe that, in general, the adjoint A^* is not necessarily densely defined, however it is readily checked that if A is a symmetric operator then, from Cauchy-Schwarz inequality, $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. Thus, if A is symmetric, then A^* is densely defined.
We have the following first criterion for self-adjointness which may be useful.

Lemma 1.2.1. *Let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined operator. Consider the graph of A:*

$$\mathbf{G}_A = \{(v, Av), v \in \mathcal{D}(A)\} \subset \mathcal{H} \oplus \mathcal{H},$$

and the complex structure

$$\begin{aligned} \mathcal{J} : &\mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H} \\ &(v, w) \to (-w, v). \end{aligned}$$

Then, the operator A is self-adjoint if and only if

$$\mathbf{G}_A^{\perp} = \mathcal{J}\left(\mathbf{G}_A\right).$$

Proof. It is checked that for any densely defined operator A

$$\mathbf{G}_{A^*} = \mathcal{J}\left(\mathbf{G}_A^{\perp}\right),$$

and the conclusion follows from routine computations. □

The following result is often useful.

Lemma 1.2.2. *Let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be an injective densely defined self-adjoint operator. Let us denote by $\mathcal{R}(A)$ the range of A. The inverse operator*

$$A^{-1} : \mathcal{R}(A) \to \mathcal{H}$$

is a densely defined self-adjoint operator.

Proof. First, let us observe that

$$\mathcal{R}(A)^{\perp} = \mathbf{Ker}(A^*) = \mathbf{Ker}(A) = \{0\}.$$

Therefore $\mathcal{R}(A)$ is dense in $\mathcal{H}$ and A^{-1} is densely defined. Now,

$$\begin{aligned} \mathbf{G}_{A^{-1}}^{\perp} &= \mathcal{J}\left(\mathbf{G}_{-A}\right)^{\perp} \\ &= \mathcal{J}\left(\mathbf{G}_{-A}^{\perp}\right) \\ &= \mathcal{J}\mathcal{J}\left(\mathbf{G}_{-A}\right) \\ &= \mathcal{J}\left(\mathbf{G}_{A^{-1}}\right). \end{aligned}$$

□

A major result in functional analysis is the spectral theorem.

Theorem 1.2.3 (Spectral theorem)**.** *Let A be a non negative self-adjoint operator on $\mathcal{H}$. There is a measure space (Ω, ν), a unitary map $U : L^2(\Omega, \nu) \to \mathcal{H}$ and a non negative real valued measurable function λ on Ω such that*

$$U^{-1}AUf(x) = \lambda(x)f(x),$$

for $x \in \Omega$, $Uf \in \mathcal{D}(A)$. Moreover, given $f \in L^2(\Omega, \nu)$, Uf belongs to $\mathcal{D}(A)$ if only if $\int_\Omega \lambda^2 f^2 d\nu < +\infty$.

Definition 1.2.4. *(Functional calculus) Let A be a non negative self-adjoint operator on $\mathcal{H}$. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a Borel function. With the notations of the spectral theorem, one defines the operator $g(A)$ by the requirement*

$$U^{-1}g(A)Uf(x) = g(\lambda(x))f(x),$$

with $\mathcal{D}(g(A)) = \{Uf, (g \circ \lambda)f \in L^2(\Omega, \nu)\}$.

Exercise 1.2.5. *Show that if A is a non negative self-adjoint operator on $\mathcal{H}$ and g is a bounded Borel function, then $g(A)$ is a bounded operator on $\mathcal{H}$.*

After those preliminaries, we turn to the study of semigroups in Hilbert spaces:

Definition 1.2.6. *A strongly continuous self-adjoint contraction semigroup is a family of self-adjoint operators $(P_t)_{t\geq 0} : \mathcal{H} \to \mathcal{H}$ everywhere defined on $\mathcal{H}$ such that:*

1. *For $s, t \geq 0$, $P_t \circ P_s = P_{s+t}$ (semigroup property);*

2. *For every $f \in \mathcal{H}$, $\lim_{t\to 0} P_t f = f$ (strong continuity);*

3. *For every $f \in \mathcal{H}$ and $t \geq 0$, $\|P_t f\| \leq \|f\|$ (contraction property).*

Definition 1.2.7. *A closed symmetric non negative bilinear form on $\mathcal{H}$ is a densely defined non negative quadratic form $\mathcal{E} : \mathcal{F} := \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ such that $\mathcal{F}$ equipped with the norm*

$$\|f\|_{\mathcal{F}}^2 = \|f\|^2 + \mathcal{E}(f)$$

is a Hilbert space. If $\mathcal{E}$ is a closed symmetric non negative bilinear form on $\mathcal{H}$, one can define for $f, g \in \mathcal{F}$, $\mathcal{E}(f, g) = \frac{1}{4}(\mathcal{E}(f + g) - \mathcal{E}(f - g))$.

One has the following theorems:

Theorem 1.2.8. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$. Then its generator A is a densely defined non positive self-adjoint operator on $\mathcal{H}$. Conversely, if A is a densely defined non positive self-adjoint operator on $\mathcal{H}$, then it is the generator a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$.*

Proof. Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$ with generator A. As we proved in the previous lecture, one has for $\lambda > 0$

$$\int_0^{+\infty} e^{-\lambda t} P_t dt = (\lambda \mathbf{Id} - A)^{-1}.$$

However, the operator $\int_0^{+\infty} e^{-\lambda t} P_t dt$ is seen to be self-adjoint, thus $(\lambda \mathbf{Id} - A)^{-1}$ is. From previous lemma, we deduce that $\lambda \mathbf{Id} - A$ is self-adjoint, from which we deduce that A is self-adjoint (exercise !).
On the other hand, let A be a densely defined non positive self-adjoint operator on $\mathcal{H}$. From spectral theorem, there is a measure space (Ω, ν), a unitary map $U : L^2(\Omega, \nu) \to \mathcal{H}$ and a non negative real valued measurable function λ on Ω such that

$$U^{-1} A U f(x) = -\lambda(x) f(x),$$

for $x \in \Omega$, $Uf \in \mathcal{D}(A)$. We define then $P_t : \mathcal{H} \to \mathcal{H}$ such that

$$U^{-1} P_t U f(x) = e^{-t\lambda(x)} f(x),$$

and let as an exercise the proof that $(P_t)_{t\geq 0}$ is a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$ with generator A. □

Theorem 1.2.9. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$. One can define a closed symmetric non negative bilinear form on $\mathcal{H}$ by*

$$\mathcal{E}(f) = \lim_{t\to 0} \left\langle \frac{\mathbf{Id}_{\mathcal{H}} - P_t}{t} f, f \right\rangle.$$

The domain $\mathcal{F}$ of this form is the set of f's for which the limit exists.

Proof. Let A be the generator of the semigroup $(P_t)_{t\geq 0}$. We use spectral theorem to represent A as

$$U^{-1} A U g(x) = -\lambda(x) g(x),$$

so that

$$U^{-1} P_t U g(x) = e^{-t\lambda(x)} g(x).$$

We then note that for every $g \in L^2(\Omega, \nu)$,

$$\left\langle \frac{\mathbf{Id}_{\mathcal{H}} - P_t}{t} Ug, Ug \right\rangle = \int_\Omega \frac{1 - e^{-t\lambda(x)}}{t} g(x)^2 d\nu(x).$$

This proves that for every $f \in \mathcal{H}$, the map $t \to \left\langle \frac{\mathbf{Id}_{\mathcal{H}} - P_t}{t} f, f \right\rangle$ is non increasing. Therefore, the limit $\lim_{t\to 0} \left\langle \frac{\mathbf{Id}_{\mathcal{H}} - P_t}{t} f, f \right\rangle$ exists if and only if $\int_\Omega (U^{-1} f)^2(x) \lambda(x) d\nu(x) < +\infty$, which is equivalent to the fact that $f \in \mathcal{D}((-A)^{1/2})$. In which case we have

$$\lim_{t\to 0} \left\langle \frac{\mathbf{Id}_{\mathcal{H}} - P_t}{t} f, f \right\rangle = \|(-A)^{1/2} f\|^2.$$

Since $(-A)^{1/2}$ is a densely defined self-adjoint operator, the quadratic form

$$\mathcal{E}(f) := \|(-A)^{1/2} f\|^2$$

is closed and densely defined on $\mathcal{F} := \mathcal{D}((-A)^{1/2})$. □

Theorem 1.2.10. *If $\mathcal{E}$ is a closed symmetric non negative bilinear form on $\mathcal{H}$. There exists a unique densely defined non positive self-adjoint operator A on $\mathcal{H}$ defined by*

$$\mathcal{D}(A) = \{f \in \mathcal{F}, \exists g \in \mathcal{H}, \forall h \in \mathcal{F}, \mathcal{E}(f,h) = -\langle h, g\rangle\}$$

$$Af = g.$$

The operator A is called the generator of $\mathcal{E}$. Conversely, if A is a densely defined non positive self-adjoint operator on $\mathcal{H}$, one can define a closed symmetric non negative bilinear form $\mathcal{E}$ on $\mathcal{H}$ by

$$\mathcal{F} = \mathcal{D}((-A)^{1/2}), \quad \mathcal{E}(f) = \|(-A)^{1/2} f\|^2.$$

Proof. Let $\mathcal{E}$ be a closed symmetric non negative bilinear form on $\mathcal{H}$. As usual, we denote by $\mathcal{F}$ the domain of $\mathcal{E}$. We note that for $\lambda > 0$, $\mathcal{F}$ equipped with the norm $(\|f\|^2 + \lambda \mathcal{E}(f))^{1/2}$ is a Hilbert space because $\mathcal{E}$ is closed. From the Riesz representation theorem, there exists then a linear operator $\mathbf{R}_\lambda : \mathcal{H} \to \mathcal{F}$ such that for every $f \in \mathcal{H}, g \in \mathcal{F}$

$$\langle f, g\rangle = \lambda \langle \mathbf{R}_\lambda f, g\rangle + \mathcal{E}(\mathbf{R}_\lambda f, g).$$

From the definition, the following properties are then easily checked:

1. $\|\mathbf{R}_\lambda f\| \le \frac{1}{\lambda}\|f\|$ (apply the definition of $\mathbf{R}_\lambda$ with $g = \mathbf{R}_\lambda f$ and then use the Cauchy-Schwarz inequality);
2. For every $f, g \in \mathcal{H}$, $\langle \mathbf{R}_\lambda f, g\rangle = \langle f, \mathbf{R}_\lambda g\rangle$;
3. $\mathbf{R}_{\lambda_1} - \mathbf{R}_{\lambda_2} + (\lambda_1 - \lambda_2)\mathbf{R}_{\lambda_1}\mathbf{R}_{\lambda_2} = 0$;
4. For every $f \in \mathcal{H}$, $\lim_{\lambda \to +\infty} \|\lambda \mathbf{R}_\lambda f - f\| = 0$.

We then claim that $\mathbf{R}_\lambda$ is invertible. Indeed, if $\mathbf{R}_\lambda f = 0$, then for $\alpha > \lambda$, one has from 3, $\mathbf{R}_\alpha f = 0$. Therefore $f = \lim_{\alpha \to +\infty} \mathbf{R}_\alpha f = 0$. Denote then

$$Af = \lambda f - \mathbf{R}_\lambda^{-1} f,$$

and $\mathcal{D}(A)$ is the range of $\mathbf{R}_\lambda$. It is straightforward to check that A does not depend on λ. The operator A is a densely defined self-adjoint operator that satisfies the properties stated in the theorem (Exercise !). □

Exercise 1.2.11. *Prove the properties 1,2,3,4 of the previous proof.*

As a conclusion, one has bijections between the set of non positive self-adjoint operators, the set of closed symmetric non negative bilinear form and the set of strongly continuous self-adjoint contraction semigroups. This is the golden triangle of the theory of heat semigroups on Hilbert spaces !

1.3 Friedrichs extension, Essential self-adjointness

As in the previous lecture, let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

Definition 1.3.1. *Let $A : \mathcal{D}(A) \to \mathcal{H}$ be a densely defined operator. A densely defined operator $\bar{A}$ is called an extension of A if $\mathcal{D}(A) \subset \mathcal{D}(\bar{A})$ and for every $f \in \mathcal{D}(A)$, $\bar{A}f = Af$.*

Theorem 1.3.2 (Friedrichs extension)**.** *Let A be a densely defined non positive symmetric operator on $\mathcal{H}$. There exists at least one self-adjoint extension of A.*

Proof. On $\mathcal{D}(A)$, let us consider the following norm

$$\|f\|_A^2 = \|f\|^2 - \langle Af, f \rangle.$$

By completing $\mathcal{D}(A)$ with respect to this norm, we get an abstract Hilbert space $(\mathcal{H}_A, \langle \cdot, \cdot \rangle_A)$. Since for $f \in \mathcal{D}(A)$, $\|f\| \le \|f\|_A$, the injection map $\iota : (\mathcal{D}(A), \|\cdot\|_A) \to (\mathcal{H}, \|\cdot\|)$ is continuous and it may therefore be extended into a continuous map $\bar{\iota} : (\mathcal{H}_A, \|\cdot\|_A) \to (\mathcal{H}, \|\cdot\|)$. Let us show that $\bar{\iota}$ is injective so that $\mathcal{H}_A$ may be identified with a subspace of $\mathcal{H}$. So, let $f \in \mathcal{H}_A$ such that $\bar{\iota}(f) = 0$. We can find a sequence $f_n \in \mathcal{D}(A)$, such that $\|f_n - f\|_A \to 0$ and $\|f_n\| \to 0$. We have then

$$\begin{aligned} \|f\|_A &= \lim_{m,n \to +\infty} \langle f_n, f_m \rangle_A \\ &= \lim_{m \to +\infty} \lim_{n \to +\infty} \langle f_n, f_m \rangle - \langle Af_n, f_m \rangle \\ &= 0, \end{aligned}$$

thus $f = 0$ and $\bar{\iota}$ is injective. Therefore, $\mathcal{H}_A$ may be identified with a subspace of $\mathcal{H}$. Since $\mathcal{D}(A) \subset \mathcal{H}_A$, one has that $\mathcal{H}_A$ is dense in $\mathcal{H}$. We consider now the quadratic form on $\mathcal{H}$ defined by

$$\mathcal{E}(f) = \|f\|_A^2 - \|f\|^2, \quad f \in \mathcal{H}_A$$

It is closed because $(\mathcal{H}_A, \langle \cdot, \cdot \rangle_A)$ is a Hilbert space. The generator of this quadratic form is then a self-adjoint extension of A. □

Remark 1.3.3. *In general self-adjoint extensions of a given symmetric operator are not unique. The operator constructed in the proof above is called the Friedrichs extension of A. It is the minimal self-adjoint extension of A.*

Definition 1.3.4. *Let A be a densely defined non positive symmetric operator on $\mathcal{H}$. We say that A is essentially self-adjoint if it admits a unique self-adjoint extension.*

We have the following convenient criterion for essential self-adjointness whose proof is let as an exercise to the reader.

Lemma 1.3.5. *Let A be a densely defined non positive symmetric operator on $\mathcal{H}$. If for some $\lambda > 0$,*

$$\mathbf{Ker}(-A^* + \lambda \mathbf{Id}) = \{0\},$$

then the operator A is essentially self-adjoint.

1.4 Diffusion operators on $\mathbb{R}^n$

Throughout the section, we consider a second order differential operator that can be written

$$L = \sum_{i,j=1}^{n} \sigma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},$$

where b_i and σ_{ij} are continuous functions on $\mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, the matrix $(\sigma_{ij}(x))_{1\le i,j\le n}$ is a symmetric and non negative matrix. Such operator is called a diffusion operator.

We will assume that there is Borel measure μ which is equivalent to the Lebesgue measure and that symmetrizes L in the sense that for every smooth and compactly supported functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} gLf d\mu = \int_{\mathbb{R}^n} fLg d\mu.$$

In what follows, as usual, we denote by $C_0^\infty(\mathbb{R}^n)$ the set of smooth and compactly supported functions $f : \mathbb{R}^n \to \mathbb{R}$.

Exercise 1.4.1. *On $C_0^\infty(\mathbb{R}^n)$, let us consider the operator*

$$L = \Delta + \langle \nabla U, \nabla \cdot \rangle,$$

where $U : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function. Show that L is symmetric with respect to the measure

$$\mu(dx) = e^{U(x)} dx.$$

Exercise 1.4.2 (Divergence form operator)**.** *On $C_0^\infty(\mathbb{R}^n)$, let us consider the operator*

$$Lf = \mathbf{div}(\sigma \nabla f),$$

where **div** *is the divergence operator defined on a C^1 function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$\mathbf{div}\ \phi = \sum_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i}$$

and where σ is a C^1 field of non negative and symmetric matrices. Show that L is a diffusion operator which is symmetric with respect to the Lebesgue measure of $\mathbb{R}^n$.

For every smooth functions $f, g : \mathbb{R}^n \to \mathbb{R}$, let us define the so-called *carré du champ*[2], which is the symmetric first-order differential form defined by:

$$\Gamma(f, g) = \frac{1}{2} \left(L(fg) - fLg - gLf \right).$$

[2] The litteral translation from French is *square of the field.*

A straightforward computation shows that

$$\Gamma(f,g)=\sum_{i,j=1}^{n}\sigma_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j},$$

so that for every smooth function f,

$$\Gamma(f,f)\geq 0.$$

Exercise 1.4.3.

1. *Show that if $f,g:\mathbb{R}^n\rightarrow\mathbb{R}$ are C^1 functions and $\phi_1,\phi_2:\mathbb{R}\rightarrow\mathbb{R}$ are also C^1 then,*

$$\Gamma(\phi_1(f),\phi_2(g))=\phi_1'(f)\phi_2'(g)\Gamma(f,g).$$

2. *Show that if $f:\mathbb{R}^n\rightarrow\mathbb{R}$ is a C^2 function and $\phi:\mathbb{R}\rightarrow\mathbb{R}$ is also C^2,*

$$L\phi(f)=\phi'(f)Lf+\phi''(f)\Gamma(f,f).$$

In the sequel we shall consider the bilinear form given for $f,g\in C_0^\infty(\mathbb{R}^n)$ by

$$\mathcal{E}(f,g)=\int_{\mathbb{R}^n}\Gamma(f,g)d\mu.$$

This is the quadratic associated to L. It is readily checked that $\mathcal{E}$ is symmetric:

$$\mathcal{E}(f,g)=\mathcal{E}(g,f),$$

and non negative

$$\mathcal{E}(f,f)\geq 0.$$

We may observe that thanks to symmetry of L,

$$\mathcal{E}(f,g)=-\int_{\mathbb{R}^n}fLgd\mu=-\int_{\mathbb{R}^n}gLfd\mu.$$

The operator L on its domain $\mathcal{D}(L)=C_0^\infty(\mathbb{R}^n)$ is a densely defined non positive symmetric operator on the Hilbert space $L^2(\mathbb{R}^n,\mu)$. However, it is not self-adjoint (why?).
The following proposition provides a useful sufficient condition for essential self-adjointness that is easy to check for several diffusion operators. We recall that a diffusion operator is said to be elliptic if the matrix σ is invertible.

Proposition 1.4.4. *If the diffusion operator L is elliptic with smooth coefficients and if there exists an increasing sequence $h_n\in C_0^\infty(\mathbb{R}^n)$, $0\leq h_n\leq 1$, such that $h_n\nearrow 1$ on $\mathbb{R}^n$, and $||\Gamma(h_n,h_n)||_\infty\rightarrow 0$, as $n\rightarrow\infty$, then the operator L is essentially self-adjoint.*

Proof. Let $\lambda > 0$. According to the previous lemma, it is enough to check that if $L^*f = \lambda f$ with $\lambda > 0$, then $f = 0$. As it was observed above, $L^*f = \lambda f$ is equivalent to the fact that, in sense of distributions, $Lf = \lambda f$. From the hypoellipticity of L, we deduce therefore that f is a smooth function. Now, for $h \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(f, h^2 f) d\mu &= -\langle f, L(h^2 f)\rangle_{L^2(\mathbb{R}^n,\mu)} \\ &= -\langle L^* f, h^2 f\rangle_{L^2(\mathbb{R}^n,\mu)} \\ &= -\lambda\langle f, h^2 f\rangle_{L^2(\mathbb{R}^n,\mu)} \\ &= -\lambda\langle f^2, h^2\rangle_{L^2(\mathbb{R}^n,\mu)} \\ &\le 0.\end{aligned}$$

Since

$$\Gamma(f, h^2 f) = h^2\Gamma(f, f) + 2fh\Gamma(f, h),$$

we deduce that

$$\langle h^2, \Gamma(f, f)\rangle_{L^2(\mathbb{R}^n,\mu)} + 2\langle fh, \Gamma(f, h)\rangle_{L^2(\mathbb{R}^n,\mu)} \le 0.$$

Therefore, by Cauchy-Schwarz inequality

$$\langle h^2, \Gamma(f, f)\rangle_{L^2(\mathbb{R}^n,\mu)} \le 4\|f|_2^2\|\Gamma(h, h)\|_\infty.$$

If we now use the sequence h_n and let $n \to \infty$, we obtain $\Gamma(f, f) = 0$ and therefore $f = 0$, as desired. □

Exercise 1.4.5. *Let*

$$L = \Delta + \langle \nabla U, \nabla \cdot\rangle,$$

where U is a smooth function on $\mathbb{R}^n$. Show that with respect to the measure $\mu(dx) = e^{U(x)}dx$, the operator L is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

Exercise 1.4.6. *On $\mathbb{R}^n$, we consider the divergence form operator*

$$Lf = \mathbf{div}(\sigma\nabla f),$$

where σ is a smooth field of positive and symmetric matrices that satisfies

$$a\|x\|^2 \le \langle x, \sigma x\rangle \le b\|x\|^2, \quad x \in \mathbb{R}^n,$$

for some constant $0 < a \le b$. Show that with respect to the Lebesgue measure, the operator L is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$

Exercise 1.4.7. *On $\mathbb{R}^n$, we consider the Schrödinger type operator*

$$H = L - V,$$

where L is a diffusion operator and $V : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. We denote

$$\Gamma(f,g) = \frac{1}{2}\left(L(fg) - fLg - gLf\right).$$

Show that if there exists an increasing sequence $h_n \in C_0^\infty(\mathbb{R}^n)$, $0 \le h_n \le 1$, such that $h_n \nearrow 1$ on $\mathbb{R}^n$, and $||\Gamma(h_n,h_n)||_\infty \to 0$, as $n \to \infty$ and that if V is bounded from below then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

Chapter 2

Markovian semigroups and Dirichlet forms

Let $(X,\mathcal{B})$ be a measurable space. We say that $(X,\mathcal{B})$ is a *good* measurable space if there is a countable family generating $\mathcal{B}$ and if every finite measure γ on $(X\times X,\mathcal{B}\otimes\mathcal{B})$ can be decomposed as

$$\gamma(dxdy)=k(x,dy)\gamma_1(dx)$$

where γ_1 is the projection of γ on the first coordinate and k is a kernel, i.e $k(x,\cdot)$ is a finite measure on $(X,\mathcal{B})$ and $x\to k(x,A)$ is measurable for every $A\in\mathcal{B}$.
For instance, if X is a Polish space (or a Radon space) equipped with its Borel σ-field, then it is a good measurable space.
Throughout the chapter, we will consider $(X,\mathcal{B},\mu)$ to be a good measurable space equipped with a σ-finite measure μ.

2.1 Markovian semigroups

Definition 2.1.1. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $L^2(X,\mu)$. The semigroup $(P_t)_{t\geq 0}$ is called Markovian if and only if for every $f\in L^2(X,\mu)$ and $t\geq 0$:*

1. $$f\geq 0,\ a.e\implies P_tf\geq 0,\ a.e\ .$$

2. $$f\leq 1,\ a.e\implies P_tf\leq 1,\ a.e\ .$$

We note that if $(P_t)_{t\geq 0}$ is Markovian, then for every $f\in L^2(X,\mu)\cap L^\infty(X,\mu)$,

$$\|P_tf\|_{L^\infty(X,\mu)}\leq\|f\|_{L^\infty(X,\mu)}.$$

As a consequence $(P_t)_{t\geq 0}$ can be extended to a contraction semigroup defined on all of $L^\infty(X,\mu)$.

Definition 2.1.2. *A transition function $\{p_t, t \geq 0\}$ on X is a family of kernels*

$$p_t : X \times \mathcal{B} \to [0, 1]$$

such that:

1. *For $t \geq 0$ and $x \in X$, $p_t(x, \cdot)$ is a finite measure on X;*
2. *For $t \geq 0$ and $A \in \mathcal{B}$ the application $x \to p_t(x, A)$ is measurable;*
3. *For $s, t \geq 0$, a.e. $x \in X$ and $A \in \mathcal{B}$,*

$$p_{t+s}(x, A) = \int_X p_t(y, A) p_s(x, dy). \tag{2.1.1}$$

The relation (2.1.1) is often called the Chapman-Kolmogorov relation

Theorem 2.1.3 (Heat kernel measure)**.** *Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. There exists a transition function $\{p_t, t \geq 0\}$ on X such that for every $f \in L^\infty(X, \mu)$ and a.e. $x \in X$*

$$P_t f(x) = \int_X f(y) p_t(x, dy), \quad t > 0. \tag{2.1.2}$$

This transition function is called the heat kernel measure associated to $(P_t)_{t \geq 0}$.

The proof relies on the following lemma sometimes called the bi-measure theorem. A set function $\nu : \mathcal{B} \otimes \mathcal{B} \to [0, +\infty)$ is called a bi-measure, if for every $A \in \mathcal{B}$, $\nu(A, \cdot)$ and $\nu(\cdot, A)$ are measures.

Lemma 2.1.4. *If $\nu : \mathcal{B} \otimes \mathcal{B} \to [0, +\infty)$ is a bi-measure, then there exists a measure γ on $\mathcal{B} \otimes \mathcal{B}$ such that for every $A, B \in \mathcal{B}$,*

$$\gamma(A \times B) = \nu(A, B).$$

Proof of Theorem 2.1.3. We assume that μ is finite and let as an exercise the extension to σ-finite measures. For $t > 0$, we consider the set function

$$\nu_t(A, B) = \int_X 1_A P_t 1_B d\mu.$$

Since P_t is supposed to be Markovian, it is a bi-measure. From the bi-measure theorem, there exists a measure γ_t on $\mathcal{B} \otimes \mathcal{B}$ such that for every $A, B \in \mathcal{B}$,

$$\gamma_t(A \times B) = \nu_t(A, B) = \int_X 1_A P_t 1_B d\mu.$$

The projection of γ_t on the first coordinate is $(P_t 1) d\mu$, thus from the measure decomposition theorem, γ_t can be decomposed as

$$\gamma_t(dxdy) = p_t(x, dy)\mu(dx)$$

for some kernel p_t. One has then for every $A, B \in \mathcal{B}$

$$\int_X 1_A P_t 1_B d\mu = \int_A \int_B p_t(x, dy)\mu(dx),$$

from which it follows that for every $f \in L^\infty(X, \mu)$, and a.e. $x \in X$

$$P_t f(x) = \int_X f(y) p_t(x, dy).$$

The relation

$$p_{t+s}(x, A) = \int_X p_t(y, A) p_s(x, dy)$$

follows from the semigroup property. □

Exercise 2.1.5. *Prove Theorem 2.1.3 if μ is σ-finite.*

Exercise 2.1.6. *Show that for every non-negative measurable function $F : X \times X \to \mathbb{R}$,*

$$\int_X \int_X F(x, y) p_t(x, dy) d\mu(x) = \int_X \int_X F(x, y) p_t(y, dx) d\mu(y). \tag{2.1.3}$$

Definition 2.1.7. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. We say that the semigroup $\{P_t\}_{t\in[0,\infty)}$ admits a heat kernel if the heat kernel measures have a density with respect to μ, i.e. there exists a measurable function $p : \mathbb{R}_{>0} \times X \times X \to \mathbb{R}_{\geq 0}$, such that for every $t > 0, a.e. x, y \in X,\ f \in L^\infty(X, \mu)$,*

$$P_t f(x) = \int_X p_t(x, y) f(y) d\mu(y).$$

If the heat kernel exists, we will often denote $p(t, x, y)$ as $p_t(x, y)$ for $t > 0$ and a.e. $x, y \in X$.

2.2 Dirichlet forms

Definition 2.2.1. *A function v on X is called a normal contraction of the function u if for almost every $x, y \in X$,*

$$|v(x) - v(y)| \leq |u(x) - u(y)| \text{ and } |v(x)| \leq |u(x)|.$$

Definition 2.2.2. *Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a densely defined closed symmetric form on $L^2(X, \mu)$. The form $\mathcal{E}$ is called a Dirichlet form if it is Markovian, that is, has the property that if $u \in \mathcal{F}$ and v is a normal contraction of u then $v \in \mathcal{F}$ and*

$$\mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

The main theorem is the following.

Theorem 2.2.3. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $L^2(X,\mu)$. Then, $(P_t)_{t\geq 0}$ is a Markovian semigroup if and only if the associated closed symmetric form on $L^2(X,\mu)$ is a Dirichlet form.*

Proof. Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X,\mu)$. There exists a transition function $\{p_t, t \geq 0\}$ on X such that for every $u \in L^\infty(X,\mu)$ and a.e. $x \in X$

$$P_t u(x) = \int_X u(y) p_t(x, dy), \quad t > 0.$$

Denote

$$k_t(x) = P_t 1(x) = \int_X p_t(x, dy).$$

We observe that from the Markovian property of P_t, we have $0 \leq k_t \leq 1$ a.e. We have then

$$\frac{1}{2}\int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x) = \int_X u(x)^2 k_t(x) d\mu(x) - \int_X u(x) P_t u(x) d\mu(x).$$

Therefore,

$$\langle u - P_t u, u \rangle = \frac{1}{2}\int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x) + \int_X u(x)^2 (1 - k_t(x)) d\mu(x).$$

Let us now assume that $u \in \mathcal{F}$ and that v is a normal contraction of u. One has

$$\int_X \int_X (v(x) - v(y))^2 p_t(x, dy) d\mu(x) \leq \int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x)$$

and

$$\int_X v(x)^2 (1 - k_t(x)) d\mu(x) \leq \int_X u(x)^2 (1 - k_t(x)) d\mu(x).$$

Therefore,

$$\langle v - P_t v, v \rangle \leq \langle u - P_t u, u \rangle$$

Since $u \in \mathcal{F}$, one knows that $\frac{1}{t}\langle u - P_t u, u \rangle$ converges to $\mathcal{E}(u)$ when $t \to 0$. Since $\frac{1}{t}\langle v - P_t v, v \rangle$ is non-increasing and bounded it does converge when $t \to 0$. Thus $v \in \mathcal{F}$ and

$$\mathcal{E}(v) \leq \mathcal{E}(u).$$

One concludes that $\mathcal{E}$ is Markovian.

Now, consider a Dirichlet form $\mathcal{E}$ and denote by P_t the associated semigroup in $L^2(X,\mu)$ and by A its generator. The main idea is to first prove that for $\lambda > 0$, the resolvent operator $(\lambda \mathbf{Id} - A)^{-1}$ preserves the positivity of function. Then, we may conclude by the fact that for $f \in L^2(X,\mu)$, in the $L^2(X,\mu)$ sense

$$P_t f = \lim_{n \to +\infty} \left(\mathbf{Id} - \frac{t}{n} L \right)^{-n} f.$$

Let $\lambda > 0$. We consider on $\mathcal{F}$ the norm

$$\|f\|_\lambda^2 = \|f\|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(f,f)$$

From the Markovian property of $\mathcal{E}$, if $u \in \mathcal{F}$, then $|u| \in \mathcal{F}$ and

$$\mathcal{E}(|u|,|u|) \le \mathcal{E}(u,u). \tag{2.2.1}$$

We consider the bounded operator

$$\mathbf{R}_\lambda = (\mathbf{Id} - \lambda A)^{-1}$$

that goes from $L^2(X,\mu)$ to $\mathcal{D}(A) \subset \mathcal{F}$. For $f \in \mathcal{F}$ and $g \in L^2(X,\mu)$ with $g \ge 0$, we have

$$\begin{aligned}
\langle |f|, \mathbf{R}_\lambda g\rangle_\lambda &= \langle |f|, \mathbf{R}_\lambda g\rangle_{L^2(X,\mu)} - \lambda \langle |f|, A\mathbf{R}_\lambda g\rangle_{L^2(X,\mu)} \\
&= \langle |f|, (\mathbf{Id} - \lambda A)\mathbf{R}_\lambda g\rangle_{L^2(X,\mu)} \\
&= \langle |f|, g\rangle_{L^2(X,\mu)} \\
&\ge |\langle f, g\rangle_{L^2(X,\mu)}| \\
&\ge |\langle f, \mathbf{R}_\lambda g\rangle_\lambda|.
\end{aligned}$$

Moreover, from inequality (2.2.1), for $f \in \mathcal{F}$,

$$\begin{aligned}
\|\ |f|\ \|_\lambda^2 &= \|\ |f|\ \|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(|f|,|f|) \\
&\le \|f\|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(f,f) \\
&\le \|f\|_\lambda^2.
\end{aligned}$$

By taking $f = \mathbf{R}_\lambda g$ in the two above sets of inequalities, we draw the conclusion

$$|\langle \mathbf{R}_\lambda g, \mathbf{R}_\lambda g\rangle_\lambda| \le \langle |\mathbf{R}_\lambda g|, \mathbf{R}_\lambda g\rangle_\lambda \le \|\ |\mathbf{R}_\lambda g|\ \|_\lambda \|\mathbf{R}_\lambda g\|_\lambda \le |\langle \mathbf{R}_\lambda g, \mathbf{R}_\lambda g\rangle_\lambda|.$$

The above inequalities are therefore equalities which implies

$$\mathbf{R}_\lambda g = |\mathbf{R}_\lambda g|.$$

As a conclusion if $g \in L^2(X,\mu)$ is a.e. ≥ 0, then for every $\lambda > 0$, $(\mathbf{Id} - \lambda A)^{-1} g \ge 0$ a.e.. Thanks to the spectral theorem, in $L^2(X,\mu)$,

$$P_t g = \lim_{n \to +\infty} \left(\mathbf{Id} - \frac{t}{n} A\right)^{-n} g.$$

By passing to a subsequence that converges pointwise almost surely, we deduce that $P_t g \ge 0$ almost surely. The proof of

$$f \le 1, \text{ a.e } \implies P_t f \le 1, \text{ a.e }.$$

follows the same lines:

- The first step is to observe that if $0 \leq f \in \mathcal{F}$, then $1 \wedge f$ and moreover

$$\mathcal{E}(1 \wedge f, 1 \wedge f) \leq \mathcal{E}(f, f).$$

- Let $f \in L^2(X, \mu)$ satisfy $0 \leq f \leq 1$ and set $g = \mathbf{R}_\lambda f = (\mathbf{Id} - \lambda A)^{-1} f \in \mathcal{F}$ and $h = 1 \wedge g$. According to the first step, $h \in \mathcal{F}$ and $\mathcal{E}(h, h) \leq \mathcal{E}(g, g)$. Now, we observe that:

$$\begin{aligned}
&\|g - h\|_\lambda^2 \\
=&\|g\|_\lambda^2 - 2\langle g, h\rangle_\lambda + \|h\|_\lambda^2 \\
=&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - 2\langle f, h\rangle_{L^2(X,\mu)} + \|h\|_{L^2(X,\mu)}^2 + \lambda\mathcal{E}(h, h) \\
=&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - \|f\|_{L^2(X,\mu)}^2 + \|f - h\|_{L^2(X,\mu)}^2 + \lambda\mathcal{E}(h, h) \\
\leq&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - \|f\|_{L^2(X,\mu)}^2 + \|f - g\|_{L^2(X,\mu)}^2 + \lambda\mathcal{E}(g, g) = 0.
\end{aligned}$$

  As a consequence $g = h$, that is $0 \leq g \leq 1$.

- The previous step shows that if $f \in L^2(X, \mu)$ satisfies $0 \leq f \leq 1$ then for every $\lambda > 0$, $0 \leq (\mathbf{Id} - \lambda L)^{-1} f \leq 1$. Thanks to spectral theorem, in $L^2(X, \mu)$,

$$\mathbf{P}_t f = \lim_{n \to +\infty} \left(\mathbf{Id} - \frac{t}{n} L\right)^{-n} f.$$

  By passing to a subsequence that converges pointwise almost surely, we deduce that $0 \leq \mathbf{P}_t f \leq 1$ almost surely.

□

2.3 The L^p theory of heat semigroups

Our goal, in this section, is to define, for $1 \leq p \leq +\infty$, P_t on $L^p(X, \mu)$. This may be done in a natural way by using the Riesz-Thorin interpolation theorem that we recall below.

Theorem 2.3.1 (Riesz-Thorin interpolation theorem)**.** *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $\theta \in (0, 1)$. Define $1 \leq p, q \leq \infty$ by*

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear map such that

$$T : L^{p_0} \to L^{q_0}, \quad \|T\|_{L^{p_0} \to L^{q_0}} = M_0$$

$$T : L^{p_1} \to L^{q_1}, \quad \|T\|_{L^{p_1} \to L^{q_1}} = M_1,$$

then, for every $f \in L^{p_0} \cap L^{p_1}$,

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p.$$

Hence T *extends uniquely as a bounded map from* L^p *to* L^q *with*

$$\|T\|_{L^p \to L^q} \leq M_0^{1-\theta} M_1^{\theta}.$$

Remark 2.3.2. *The statement that* T *is a linear map such that*

$$T : L^{p_0} \to L^{q_0}, \quad \|T\|_{L^{p_0} \to L^{q_0}} = M_0$$

$$T : L^{p_1} \to L^{q_1}, \quad \|T\|_{L^{p_1} \to L^{q_1}} = M_1,$$

means that there exists a map $T : L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ *with*

$$\sup_{f \in L^{p_0} \cap L^{p_1}, \|f\|_{p_0} \leq 1} \|Tf\|_{q_0} = M_0$$

and

$$\sup_{f \in L^{p_0} \cap L^{p_1}, \|f\|_{p_1} \leq 1} \|Tf\|_{q_1} = M_1.$$

In such a case, T *can be uniquely extended to bounded linear maps* $T_0 : L^{p_0} \to L^{q_0}$ *,* $T_1 : L^{p_1} \to L^{q_1}$*. With a slight abuse of notation, these two maps are both denoted by* T *in the theorem.*

Remark 2.3.3. *If* $f \in L^{p_0} \cap L^{p_1}$ *and* p *is defined by* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$*, then by Hölder's inequality,* $f \in L^p$ *and*

$$\|f\|_p \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$

We now are in position to state the following theorem:

Theorem 2.3.4. *Let* $(P_t)_{t \geq 0}$ *be a strongly continuous self-adjoint contraction Markovian semigroup on* $L^2(X, \mu)$*. The space* $L^1 \cap L^\infty$ *is invariant under* P_t *and* P_t *may be extended from* $L^1 \cap L^\infty$ *to a contraction semigroup* $(P_t^{(p)})_{t \geq 0}$ *on* L^p *for all* $1 \leq p \leq \infty$*: For* $f \in L^p$,

$$\|P_t f\|_{L^p} \leq \|f\|_{L^p}.$$

These semigroups are consistent in the sense that for $f \in L^p \cap L^q$,

$$P_t^{(p)} f = P_t^{(q)} f.$$

Proof. If $f, g \in L^1 \cap L^\infty$ which is a subset of $L^1 \cap L^\infty$, then,

$$\begin{aligned} \left| \int_X (P_t f) g d\mu \right| &= \left| \int_X f (P_t g) d\mu \right| \\ &\leq \|f\|_{L^1} \|P_t g\|_{L^\infty} \\ &\leq \|f\|_{L^1} \|g\|_{L^\infty}. \end{aligned}$$

This implies

$$\|P_t f\|_{L^1} \leq \|f\|_{L^1}.$$

The conclusion follows then from the Riesz-Thorin interpolation theorem. □

Exercise 2.3.5. *Show that if $f \in L^p$ and $g \in L^q$ with $\frac{1}{p}+\frac{1}{q}=1$ then,*

$$\int_{\mathbb{R}^n} f P_t^{(q)} g d\mu = \int_{\mathbb{R}^n} g P_t^{(p)} f d\mu.$$

Exercise 2.3.6.

1. *Show that for each $f \in L^1$, the L^1-valued map $t \to P_t^{(1)} f$ is continuous.*

2. *Show that for each $f \in L^p$, $1 < p < 2$, the L^p-valued map $t \to P_t^{(p)} f$ is continuous.*

3. *Finally, by using the reflexivity of L^p, show that for each $f \in L^p$ and every $p \ge 1$, the L^p-valued map $t \to P_t^{(p)} f$ is continuous.*

We mention, that in general, the L^∞ valued map $t \to P_t^{(\infty)} f$ is not continuous.

2.4 Diffusion operators as Markov operators

In this section, we consider a diffusion operator

$$L = \sum_{i,j=1}^n \sigma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial}{\partial x_i},$$

where b_i and σ_{ij} are continuous functions on $\mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, the matrix $(\sigma_{ij}(x))_{1\le i,j\le n}$ is a symmetric and non negative matrix. Our goal is to prove that if L is essentially self-adjoint, then the semigroup it generates is Markovian. We will also prove that this semigroup is solution of the heat equation associated to L.
As before, we will assume that there is Borel measure μ which is equivalent to the Lebesgue measure and that symmetrizes L in the sense that for every smooth and compactly supported functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} g L f d\mu = \int_{\mathbb{R}^n} f L g d\mu.$$

Our first goal will be to prove that if L is essentially self-adjoint, then the semigroup it generates in $L^2(\mathbb{R}^n, \mu)$ is Markovian. The key lemma is the so-called Kato inequality:

Lemma 2.4.1 (Kato inequality)**.** *Let L be a diffusion operator on $\mathbb{R}^n$ with symmetric and invariant measure μ. Let $u \in C_0^\infty(\mathbb{R}^n)$. Define*

$$\begin{aligned} \mathbf{sgn}\ u &= 0 \quad \text{if } u(x)=0, \\ &= \frac{u(x)}{|u(x)|} \quad \text{if } u(x) \neq 0. \end{aligned}$$

In the sense of distributions, we have the following inequality

$$L|u| \ge (\mathbf{sgn}\ u) L u.$$

Proof. If ϕ is a smooth and convex function and if u is assumed to be smooth, it is readily checked that

$$L\phi(u) = \phi'(u)Lu + \phi''(u)\Gamma(u,u) \geq \phi'(u)Lu.$$

By choosing for ϕ the function

$$\phi_\varepsilon(x) = \sqrt{x^2+\varepsilon^2}, \quad \varepsilon > 0,$$

we deduce that for every smooth function $u \in C_0^\infty(\mathbb{R}^n)$,

$$L\phi_\varepsilon(u) \geq \frac{u}{\sqrt{x^2+\varepsilon^2}} Lu.$$

As a consequence this inequality holds in the sense of distributions, that is for every $f \in \mathcal{C}_c(\mathbb{R}^n, \mathbb{R})$, $f \geq 0$,

$$\int_{\mathbb{R}^n} f L\phi_\varepsilon(u) d\mu \geq \int_{\mathbb{R}^n} f \frac{u}{\sqrt{u^2+\varepsilon^2}} Lu d\mu$$

Letting $\varepsilon \to 0$ gives the expected result. □

From Kato inequality, it is relatively easy to see that if L is an essentially self-adjoint diffusion operator, then the associated quadratic form is Markovian. As a consequence, we deduce the following theorem.

Proposition 2.4.2. *Let L be a diffusion operator on $\mathbb{R}^n$ with symmetric and invariant measure μ. Assume that L is essentially self-adjoint, then the semigroup it generates is Markovian.*

Next, we connect the semigroup associated to a diffusion operator L to the parabolic following Cauchy problem:

$$\frac{\partial u}{\partial t} = Lu, \quad u(0,x) = f(x).$$

In the remainder of the section, we assume that the diffusion operator L is elliptic with smooth coefficients and that there exists an increasing sequence $h_n \in C_0^\infty(\mathbb{R}^n)$, $0 \leq h_n \leq 1$, such that $h_n \nearrow 1$ on $\mathbb{R}^n$, and $||\Gamma(h_n, h_n)||_\infty \to 0$, as $n \to \infty$. In particular, we know from this assumption that the operator L is essentially self-adjoint.

Proposition 2.4.3. *Let $f \in L^p(\mathbb{R}^n, \mu)$, $1 \leq p \leq \infty$, and let*

$$u(t,x) = P_t f(x), \quad t \geq 0, x \in \mathbb{R}^n.$$

Then u is smooth on $(0,+\infty) \times \mathbb{R}^n$ and is a strong solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = Lu, \quad u(0,x) = f(x).$$

Proof. For $\phi \in C_0^\infty((0,+\infty)\times\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}}\left(\left(-\frac{\partial}{\partial t}-L\right)\phi(t,x)\right)u(t,x)d\mu(x)dt &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}\left(\left(-\frac{\partial}{\partial t}-L\right)\phi(t,x)\right)P_t f(x)dxdt\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}P_t\left(\left(-\frac{\partial}{\partial t}-L\right)\phi(t,x)\right)f(x)dxdt\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}-\frac{\partial}{\partial t}\left(P_t\phi(t,x)f(x)\right)dxdt\\ &= 0.\end{aligned}$$

Therefore u is a weak solution of the equation $\frac{\partial u}{\partial t}=Lu$. Since u is smooth it is also a strong solution. □

We now address the uniqueness of solutions.

Proposition 2.4.4. *Let $v(x,t)$ be a non negative function such that*

$$\frac{\partial v}{\partial t}\leq Lv,\quad v(x,0)=0,$$

and such that for every $t>0$,

$$\|v(\cdot,t)\|_{L^p(\mathbb{R}^n,\mu)}<+\infty,$$

where $1<p<+\infty$. Then $v(x,t)=0$.

Proof. Let $x_0\in X$ and $h\in C_0^\infty(\mathbb{R}^n)$. Since u is a subsolution with the zero initial data, for any $\tau\in(0,T)$,

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^n}h^2(x)v^{p-1}(x,t)Lv(x,t)d\mu(x)dt\\ \geq&\int_0^\tau\int_{\mathbb{R}^n}h^2(x)v^{p-1}\frac{\partial v}{\partial t}d\mu(x)dt\\ =&\frac{1}{p}\int_0^\tau\frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n}h^2(x)v^p d\mu(x)\right)dt\\ =&\frac{1}{p}\int_{\mathbb{R}^n}h^2(x)v^p(x,\tau)d\mu(x).\end{aligned}$$

On the other hand, integrating by parts yields

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{R}^n}h^2(x)v^{p-1}(x,t)Lv(x,t)d\mu(x)dt\\ =&-\int_0^\tau\int_{\mathbb{R}^n}2hv^{p-1}\Gamma(h,v)d\mu dt-\int_0^\tau\int_X h^2(p-1)v^{p-2}\Gamma(v)d\mu dt.\end{aligned}$$

Observing that

$$0 \leq \left(\sqrt{\frac{2}{p-1}\Gamma(h)}v - \sqrt{\frac{p-1}{2}\Gamma(v)}h \right)^2 \leq \frac{2}{p-1}\Gamma(h)v^2 + 2\Gamma(h,v)hv + \frac{p-1}{2}\Gamma(v)h^2,$$

we obtain the following estimate.

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^n} h^2(x)v^{p-1}(x,t)Lv(x,t)d\mu(x)dt \\
\leq &\int_0^\tau \int_{\mathbb{R}^n} \frac{2}{p-1}\Gamma(h)v^p d\mu dt - \int_0^\tau \int_{\mathbb{R}^n} \frac{p-1}{2} h^2 v^{p-2}\Gamma(v) d\mu dt \\
= &\int_0^\tau \int_{\mathbb{R}^n} \frac{2}{p-1}\Gamma(h)v^p d\mu dt - \frac{2(p-1)}{p^2}\int_0^\tau \int_{\mathbb{R}^n} h^2 \Gamma(v^{p/2}) d\mu dt.
\end{aligned}$$

Combining with the previous conclusion we obtain ,

$$\int_X h^2(x)v^p(x,\tau)d\mu(x) + \frac{2(p-1)}{p}\int_0^\tau \int_{\mathbb{R}^n} h^2\Gamma(v^{p/2})d\mu dt \leq \frac{2p}{(p-1)}\|\Gamma(h)\|_\infty^2 \int_0^\tau \int_{\mathbb{R}^n} v^p d\mu dt.$$

By using the previous inequality with an increasing sequence $h_n \in C_0^\infty(\mathbb{R}^n)$, $0 \leq h_n \leq 1$, such that $h_n \nearrow 1$ on $\mathbb{R}^n$, and $||\Gamma(h_n,h_n)||_\infty \to 0$, as $n \to \infty$, and letting $n \to +\infty$, we obtain $\int_X v^p(x,\tau)d\mu(x) = 0$ thus $v = 0$. □

As a consequence of this result, any solution in $L^p(\mathbb{R}^n,\mu)$, $1 < p < +\infty$ of the heat equation $\frac{\partial u}{\partial t} = Lu$ is uniquely determined by its initial condition, and is therefore of the form $u(t,x) = P_t f(x)$. We stress that without further conditions, this result fails when $p = 1$ or $p = +\infty$.

2.5 Sobolev inequality

In this lecture, we. study Sobolev inequalities on Dirichlet spaces. The approach we develop is related to Hardy-Littlewood-Sobolev theory
The link between the Hardy-Littlewood-Sobolev theory and heat kernel upper bounds is due to Varopoulos, but the proof I give below I learnt it from my colleague RodrigoBañuelos. It bypasses the Marcinkiewicz interpolation theorem,that was originally used by Varopoulos by using instead the Stein's maximal ergodic lemma. The advantage of the method is to get an explicit (non sharp) constant for the Sobolev inequality
Let $(X,\mathcal{B})$ be a good measurable space equipped with a σ-finite measure μ. Let $\mathcal{E}$ be a Dirichlet form on X. As usual, we denote by P_t the semigroup generated by P_t and we assume $P_t 1 = 1$.
We have the following so-called maximal ergodic lemma, which was first proved by Stein. We give here a probabilistic proof since it comes with a nice constant but you can for instance find the original (non probabilistic) proof here.

Lemma 2.5.1. *(Stein's maximal ergodic theorem) Let* $p > 1$*. For* $f \in L^p(X, \mu)$*, denote* $f^*(x) = \sup_{t \geq 0} |P_t f(x)|$*. We have* $\|f^*\|_{L^p(X,\mu)} \leq \frac{p}{p-1} \|f\|_{L^p(X,\mu)}$*.*

Proof. For $x \in X$, we denote by $(X_t^x)_{t \geq 0}$ the Markov process associated with the semigroup P_t and started at x (we assume that such process exists without commenting on the exact assumptions). We fix $T > 0$. By construction, for $t \leq T$, we have,

$$P_{T-t} f(X_T^x) = \mathbb{E}\left(f(X_{2T-t}^x) | X_T^x \right),$$

and thus

$$P_{2(T-t)} f(X_T^x) = \mathbb{E}\left((P_{T-t} f)(X_{2T-t}^x) | X_T^x \right).$$

As a consequence, we obtain

$$\sup_{0 \leq t \leq T} |P_{2(T-t)} f(X_T^x)| \leq \mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_{2T-t}^x)| \mid X_T^x \right).$$

Jensen's inequality yields then

$$\sup_{0 \leq t \leq T} |P_{2(T-t)} f(X_T^x)|^p \leq \mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_{2T-t}^x)|^p \mid X_T^x \right).$$

We deduce

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |P_{2(T-t)} f(X_T^x)|^p \right) \leq \mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_{2T-t}^x)|^p \right).$$

Integrating the inequality with respect to the measure μ, we obtain

$$\left\| \sup_{0 \leq t \leq T} |P_{2(T-t)} f| \right\|_p \leq \left(\int_X \mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_{2T-t}^x)|^p \right) d\mu(x) \right)^{1/p}.$$

By reversibility, we get then

$$\left\| \sup_{0 \leq t \leq T} |P_{2(T-t)} f| \right\|_p \leq \left(\int_X \mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_t^x)|^p \right) d\mu(x) \right)^{1/p}.$$

We now observe that the process $(P_{T-t} f)(X_t^x)$ is martingale and thus Doob's maximal inequality gives

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |(P_{T-t} f)(X_t^x)|^p \right)^{1/p} \leq \frac{p}{p-1} \mathbb{E}\left(|f(X_T^x)|^p \right)^{1/p}.$$

The proof is complete. $\square$

We now turn to the theorem by Varopoulos. In the sequel, we assume that the semigroup P_t admits a measurable heat kernel $p(x, y, t)$.

Theorem 2.5.2. *Let $n > 0$, $0 < \alpha < n$, and $1 < p < \frac{n}{\alpha}$. If there exists $C > 0$ such that for every $t > 0$, $x, y \in X$, $p(x, y, t) \leq \frac{C}{t^{n/2}}$, then for every $f \in L^p(X, \mu)$,*

$$\|(-L)^{-\alpha/2} f\|_{\frac{np}{n-p\alpha}} \leq \left(\frac{p}{p-1}\right)^{1-\alpha/n} \frac{2nC^{\alpha/n}}{\alpha(n-p\alpha)\Gamma(\alpha/2)} \|f\|_p,$$

where L is the generator of $\mathcal{E}$.

Proof. We first observe that the bound

$$p(x, y, t) \leq \frac{C}{t^{n/2}},$$

implies that

$$|P_t f(x)| \leq \frac{C^{1/p}}{t^{n/2p}} \|f\|_p$$

. Denote

$$I_\alpha f(x) = (-L)^{-\alpha/2} f(x)$$

. We have

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^{+\infty} t^{\alpha/2-1} P_t f(x) dt$$

Pick $\delta > 0$, to be later chosen, and split the integral in two parts:

$$I_\alpha f(x) = J_\alpha f(x) + K_\alpha f(x),$$

where

$$J_\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^{\delta} t^{\alpha/2-1} P_t f(x) dt$$

and

$$K_\alpha f(x) = \frac{1}{\Gamma(\alpha/2)} \int_\delta^{+\infty} t^{\alpha/2-1} P_t f(x) dt.$$

We have

$$|J_\alpha f(x)| \leq \frac{1}{\Gamma(\alpha/2)} \int_0^{+\infty} t^{\alpha/2-1} dt |f^*(x)| = \frac{2}{\alpha\Gamma(\alpha/2)} \delta^{\alpha/2} |f^*(x)|.$$

On the other hand,

$$\begin{aligned} |K_\alpha f(x)| &\leq \frac{1}{\Gamma(\alpha/2)} \int_\delta^{+\infty} t^{\alpha/2-1} |P_t f(x)| dt \\ &\leq \frac{C^{1/p}}{\Gamma(\alpha/2)} \int_\delta^{+\infty} t^{\frac{\alpha}{2}-\frac{n}{2p}-1} dt \|f\|_p \\ &\leq \frac{C^{1/p}}{\Gamma(\alpha/2)} \frac{1}{-\frac{\alpha}{2}+\frac{n}{2p}} \delta^{\frac{\alpha}{2}-\frac{n}{2p}} \|f\|_p. \end{aligned}$$

We deduce

$$|I_\alpha f(x)| \le \frac{2}{\alpha\Gamma(\alpha/2)}\delta^{\alpha/2}|f^*(x)| + \frac{C^{1/p}}{\Gamma(\alpha/2)}\frac{1}{-\frac{\alpha}{2}+\frac{n}{2p}}\delta^{\frac{\alpha}{2}-\frac{n}{2p}}\|f\|_p.$$

Optimizing the right hand side of the latter inequality with respect to δ yields

$$|I_\alpha f(x)| \le \frac{2nC^{\alpha/n}}{\alpha(n-p\alpha)\Gamma(\alpha/2)}\|f\|_p^{\alpha p/n}|f^*(x)|^{1-p\alpha/n}.$$

The proof is then completed by using Stein's maximal ergodic theorem. □

A special case, of particular interest, is when $\alpha = 1$ and $p = 2$. We get in that case the following Sobolev inequality:

Theorem 2.5.3. *Let $n > 2$. If there exists $C > 0$ such that for every $t > 0$, $x, y \in X$, $p(x, y, t) \le \frac{C}{t^{n/2}}$, then for every $f \in \mathcal{F}$,*

$$\|f\|_{\frac{2n}{n-2}} \le 2^{1-1/n}\frac{2nC^{1/n}}{(n-2)\sqrt{\pi}}\sqrt{\mathcal{E}(f)}.$$

We mention that the constant in the above Sobolev inequality is not sharp even in the Euclidean case.
In many situations, heat kernel upper bounds with a polynomial decay are only available in small times the following result is thus useful:

Theorem 2.5.4. *Let $n > 0$, $0 < \alpha < n$, and $1 < p < \frac{n}{\alpha}$. If there exists $C > 0$ such that for every $0 < t \le 1$, $x, y \in X$,*

$$p(x, y, t) \le \frac{C}{t^{n/2}},$$

then, there is constant C' such that for every $f \in L^p(X, \mu)$,

$$\|(-L+1)^{-\alpha/2}f\|_{\frac{np}{n-p\alpha}} \le C'\|f\|_p.$$

Proof. We apply the Varopoulos theorem to the semigroup $Q_t = e^{-t}P_t$. Details are let to the reader □

The following corollary shall be later used:

Corollary 2.5.5. *Let $n > 2$. If there exists $C > 0$ such that for every $0 < t \le 1$, $x, y \in X$,*

$$p(x, y, t) \le \frac{C}{t^{n/2}},$$

then there is constant C' such that for every $f \in \mathcal{F}$,

$$\|f\|_{\frac{2n}{n-2}} \le C'\left(\sqrt{\mathcal{E}(f)} + \|f\|_2\right).$$

Chapter 3

Dirichlet spaces with Gaussian or sub-Gaussian heat kernel estimates

Let (X, d, μ) be a locally compact and complete metric measure space where μ is a Radon measure supported on X. Let now $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a Dirichlet form on X. Throughout the chapter, we assume the following:

- $B(x,r) := \{y \in X \mid d(x,y) < r\}$ has compact closure for any $x \in X$ and any $r \in (0, \infty)$;

We also assume that the semigroup $\{P_t\}$ associated with $\mathcal{E}$ is stochastically complete (i.e. $P_t 1 = 1$) and has a continuous heat kernel $p_t(x,y)$ satisfying, for some $c_1, c_2, c_3, c_4 \in (0, \infty)$ and $d_H \geq 1, d_W \in [2, +\infty)$,

$$c_1 t^{-d_H/d_W} \exp\left(-c_2 \Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W - 1}}\right) \leq p_t(x,y) \leq c_3 t^{-d_H/d_W} \exp\left(-c_4 \Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W - 1}}\right) \tag{3.0.1}$$

for $\mu \times \mu$-a.e. $(x,y) \in X \times X$ and each $t \in \big(0, +\infty\big)$.
The exact values of c_1, c_2, c_3, c_4 are irrelevant in our analysis. However, the parameters d_H and d_W are important; they are metric invariant of the space. We will see that the parameter d_H is the volume growth exponent. The parameter d_W is more subtle and is called the walk dimension. When $d_W = 2$, one speaks of Gaussian estimates and when $d_W > 2$, one speaks then of sub-Gaussian estimates.

3.1 Examples

3.1.1 Uniformly elliptic divergence form diffusion operators

On $\mathbb{R}^n$, we consider the divergence form operator

$$Lf = -\mathbf{div}(\sigma \nabla f),$$

where σ is a smooth field of positive and symmetric matrices that satisfies

$$a\|x\|^2 \leq \langle x, \sigma(y)x \rangle \leq b\|x\|^2, \quad x, y \in \mathbb{R}^n,$$

for some constant $0 < a \leq b$. We know that with respect to the Lebesgue measure, the operator L is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and its self-adjoint extension (still denoted L) is the generator of the Dirichlet form

$$\mathcal{E}(f) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx, \quad f \in W^{1,2}(\mathbb{R}^n).$$

Theorem 3.1.1 (Nash, Aronson, Davies)**.** *The Dirichlet form $\mathcal{E}$ admits a heat kernel satisfying the Gaussian heat kernel estimates* (3.0.1) *with $d_W = 2$ and $d_H = n$ and the distance is the Euclidean distance.*

3.1.2 Riemannian manifolds

Let $(\mathbb{M}, g)$ be a complete n-dimensional Riemannian manifold with Riemannian volume measure μ and Riemannian distance d. We assume that the Ricci curvature of $\mathbb{M}$ is non negative. We consider the standard Dirichlet form $\mathcal{E}$ on $\mathbb{M}$, which is obtained by closing the bilinear form

$$\mathcal{E}(f, g) = \int_{\mathbb{M}} \langle \nabla f, \nabla g \rangle d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

It is well-known result by Li and Yau that the heat semigroup P_t admits a smooth heat kernel function $p_t(x, y)$ on $[0, \infty) \times \mathbb{M} \times \mathbb{M}$ for which there are constants $c_1, c_2, C > 0$ such that whenever $t > 0$ and $x, y \in X$,

$$\frac{1}{C} \frac{e^{-c_1 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \leq p_t(x, y) \leq C \frac{e^{-c_2 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

From the Bishop-Gromov comparison theorem, it is known that the non-negative Ricci curvature lower bound assumption implies

$$\mu(B(x, R)) \leq CR^n.$$

If we assume that there is a lower bound

$$\mu(B(x, R)) \geq cR^n,$$

then $p_t(x, y)$ therefore admits Gaussian heat kernel estimates ($d_H = n$, $d_W = 2$).

3.1.3 Carnot groups

A Carnot group of step N is a simply connected Lie group $\mathbb{G}$ whose Lie algebra can be stratified as follows:

$$\mathfrak{g} = \mathcal{V}_1 \oplus ... \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \text{ for } s > N.$$

From the above properties, Carnot groups are nilpotent. The number

$$Q = \sum_{i=1}^{N} i \dim \mathcal{V}_i$$

is called the homogeneous dimension of $\mathbb{G}$.
Let $V_1, ..., V_d$ be a basis of the vector space $\mathcal{V}_1$. The vectors V_i's can be seen as left invariant vector fields on $\mathbb{G}$. The left invariant sub-Laplacian on $\mathbb{G}$ is the operator:

$$L = \sum_{i=1}^{d} V_i^2.$$

It is hypoelliptic and essentially self-adjoint on the space of smooth and compactly supported function $f : \mathbb{G} \to \mathbb{R}$ with the respect to the Haar measure μ of $\mathbb{G}$. The heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{G}$, defined through the spectral theorem, is then seen to be a Markov semigroup. By hypoellipticity of L, this heat semigroup admits a heat kernel denoted by $p_t(g, g')$. It is then known that p_t satisfies the double-sided Gaussian bounds:

$$\frac{C^{-1}}{t^{Q/2}} \exp\left(-\frac{C_1 d(g, g')^2}{t}\right) \leq p_t(g, g') \leq \frac{C}{t^{Q/2}} \exp\left(-C_2 \frac{d(g, g')^2}{t}\right), \tag{3.1.1}$$

for some constants $C, C_1, C_2 > 0$. Here $d(g, g')$ denotes the Carnot-Carathéodory distance from g to g' on $\mathbb{G}$ which is defined by

$$d(g, g') = \sup \left\{ |f(g) - f(g')|, \quad \sum_{i=1}^{d} (V_i f)^2 \leq 1 \right\}.$$

In that case, we therefore have $d_H = Q$ and $d_W = 2$.

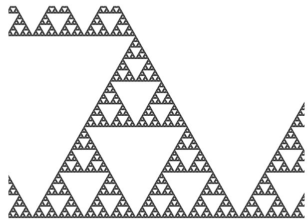

Figure 3.1: A part of an infinite, or unbounded, Sierpiński gasket.

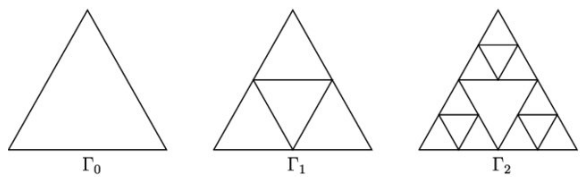

Figure 3.2: Graph approximation of the Sierpiński gasket.

3.1.4 Unbounded Sierpiński gasket

If X is the unbounded Sierpiński gasket, one considers on X the distance d induced by the Euclidean distance in $\mathbb{R}^2$. The Hausdorff measure on X is denoted by μ.

$$\mathcal{E}_n(f) := \sum_{x,y \in \Gamma_n, x \sim y} (f(x) - f(y))^2$$

$$\mathcal{E}(f) := \lim_{n \to +\infty} \left(\frac{5}{3}\right)^n \mathcal{E}_n(f)$$

$$\mathcal{F} := \left\{ f \in L^2(X, \mu), \mathcal{E}(f) < +\infty \right\}$$

Theorem 3.1.2 (Kigami, Barlow-Perkins)**.** *The quadratic form $\mathcal{E}$ with domain $\mathcal{F}$ is a Dirichlet form on $L^2(X,\mu)$ which admits a heat kernel satisfying the sub-Gaussian heat kernel estimates* (3.0.1) *with $d_W = \frac{\ln 5}{\ln 2}$ and $d_H = \frac{\ln 3}{\ln 2}$.*

3.2 Ahlfors regularity

We consider in this section the general framework outlined at the beginning of the chapter. Our goal in the section is to prove that the space (X, d, μ) is Ahlfors d_H-regular. As a consequence, the number d_H is a metric invariant.

Theorem 3.2.1. *The space (X, d, μ) is Ahlfors d_H-regular, i.e. there exist constants $c, C > 0$ such that for every $x \in X$, $R \geq 0$,*

$$cR^{d_H} \leq \mu(B(x,R)) \leq CR^{d_H}.$$

We divide onto several lemmas.

Lemma 3.2.2. *There exists a constant $C > 0$ such that for every $x \in X$, $R \geq 0$,*

$$\mu(B(x,R)) \leq CR^{d_H}.$$

Proof.

$$\begin{aligned}
&\int_X p_t(x,y)d\mu(y) \\
\geq &\int_{B(x,R)} p_t(x,y)d\mu(y) \\
\geq &c_1 t^{-d_H/d_W} \int_{B(x,R)} \exp\left(-c_2\left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)d\mu(y) \\
\geq &c_1 t^{-d_H/d_W} \exp\left(-c_2\left(\frac{R^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)\mu(B(x,R)).
\end{aligned}$$

Since $\int_X p_t(x,y)d\mu(y) = 1$, one deduces

$$\mu(B(x,R)) \leq \frac{1}{c_1} t^{d_H/d_W} \exp\left(c_2\left(\frac{R^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right).$$

Choosing $t = R^{d_W}$ yields the expected result. □

The second lemma is the following:

Lemma 3.2.3. *There exists a constant $C > 0$ such that for every $x \in X$, $t > 0$, $R > 0$,*

$$\int_{X \setminus B(x,r)} p_t(x,y)d\mu(y) \leq C \exp\left(-\frac{1}{C}\left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

Proof.

$$\begin{aligned}
&\int_{X\setminus B(x,r)} p_t(x,y)d\mu(y) && (3.2.1)\\
&\le \frac{c_3}{t^{d_H/d_W}} \int_{X\setminus B(x,r)} \exp\Big(-c_4\Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big)\, d\mu(y)\\
&= \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} \exp\Big(-c_4\Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big)\, d\mu(y)\\
&\le \frac{c_3}{t^{d_H/d_W}} \sum_{k=1}^{\infty} \mu\big(B(x,2^k r)\big) \exp\Big(-c_4\Big(\frac{2^{(k-1)d_W} r^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big)\\
&\le \frac{C}{t^{d_H/d_W}} \sum_{k=1}^{\infty} r^{d_H} 2^{kd_H} \exp\Big(-c_4\Big(\frac{r^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big(2^{\frac{d_W}{d_W-1}}\Big)^{k-1}\Big)\\
&= C\sum_{k=1}^{\infty} 2^{d_H}\Big(\frac{r^{d_W}}{t}2^{d_W(k-1)}\Big)^{d_H/d_W} \exp\Big(-2^{-\frac{d_W}{d_W-1}}c_4\Big(\frac{r^{d_W}}{t}2^{d_W k}\Big)^{\frac{1}{d_W-1}}\Big)\\
&\le C\sum_{k=1}^{\infty} \int_{(r^{d_W}/t)(2^{d_W})^{k-1}}^{(r^{d_W}/t)(2^{d_W})^{k}} s^{d_H/d_W} \exp\Big(-2^{-\frac{d_W}{d_W-1}}c_4 s^{\frac{1}{d_W-1}}\Big)\frac{1}{(d_W\log 2)s}\, ds\\
&= C\int_{r^{d_W}/t}^{\infty} s^{d_H/d_W-1} \exp\Big(-2^{-\frac{d_W}{d_W-1}}c_4 s^{\frac{1}{d_W-1}}\Big)\, ds\\
&\le C\exp\Big(-\frac{1}{C}\Big(\frac{r^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big). && (3.2.2)
\end{aligned}$$

□

The result follows then from the last following lemma:

Lemma 3.2.4. *There exists a constant $c > 0$ such that for every $x \in X$, $R \ge 0$,*

$$\mu(B(x,R)) \ge cR^{d_H}.$$

Proof. On one hand

$$\begin{aligned}
&\int_{B(x,r)} p_t(x,y)d\mu(y)\\
\le& c_3 t^{-d_H/d_W} \int_{B(x,r)} \exp\Big(-c_4\Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big)\, d\mu(y)\\
\le& c_3 t^{-d_H/d_W}\mu(B(x,r)).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_{B(x,r)} p_t(x,y)d\mu(y)\\
=&1-\int_{X\setminus B(x,r)} p_t(x,y)d\mu(y)\\
\ge&1-C\exp\left(-\frac{1}{C}\left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)
\end{aligned}$$

We conclude

$$\mu(B(x,r))\ge ct^{d_H/d_W}\left(1-C\exp\left(-\frac{1}{C}\left(\frac{r^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)\right).$$

Choosing $t=\delta r^{d_W}$ with $\delta>0$ small enough one gets the expected result. □

3.3 Domain of the form

In this section, we still consider a Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$ on a metric measure space (X,d,μ) whose heat kernel satisfies the sub-Gaussian estimates (3.0.1).
In what follows, if Λ_1 and Λ_2 are two functionals defined on a domain $\mathcal{D}$, we will write

$$\Lambda_1(f)\simeq\Lambda_2(f)$$

if there exist constants $c,C>0$ such that for every $f\in\mathcal{D}$

$$c\Lambda_1(f)\le\Lambda_2(f)\le C\Lambda_1(f).$$

Our goal is to prove the following theorem.

Theorem 3.3.1. *On* $\mathcal{F}$

$$\begin{aligned}
\mathcal{E}(f)&\simeq\liminf_{r\to0^+}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\\
&\simeq\limsup_{r\to0^+}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\\
&\simeq\sup_{0<r\le1}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\\
&\simeq\sup_{r>0}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)
\end{aligned}$$

Note that it is enough to prove that

$$c\sup_{r>0}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)\le\mathcal{E}(f)\le C\liminf_{r\to0^+}\frac{1}{r^{d_W}}\int_X\int_{B(x,r)}\frac{|f(y)-f(x)|^2}{\mu(B(x,r))}\,d\mu(y)\,d\mu(x)$$

We first prove the lower bound:

Lemma 3.3.2. *For $f \in \mathcal{F}$,*

$$c \sup_{r>0} \frac{1}{r^{d_W}} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^2}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \le \mathcal{E}(f).$$

Proof. We first prove the lower bound. For $s, t > 0$ and $\alpha > 0$,

$$\begin{aligned}
&\int_X \int_X |f(x)-f(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \\
&\quad \ge \int_X \int_{B(y,s)} |f(x)-f(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \\
&\quad \ge c_1 t^{-d_H/d_W} \int_X \int_{B(y,s)} |f(x)-f(y)|^2 \exp\Big(-c_2 \Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big) d\mu(x) d\mu(y) \\
&\quad \ge c_1 t^{-d_H/d_W} \exp\Big(-c_2 \Big(\frac{s^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big) \int_X \int_{B(y,s)} |f(x)-f(y)|^2 d\mu(x) d\mu(y).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{1}{s^{d_H+d_W}} \int_X \int_{B(y,s)} |f(x)-f(y)|^2 d\mu(x) d\mu(y) \\
&\le C \frac{t^{d_H/d_W}}{s^{d_H+d_W}} \exp\Big(c_2 \Big(\frac{s^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\Big) \int_X \int_X |f(x)-f(y)|^2 p_t(x,y) d\mu(x) d\mu(y)
\end{aligned}$$

Choosing $t = s^{d_W}$, one deduces

$$\frac{1}{s^{d_H+d_W}} \int_X \int_{B(y,s)} |f(x)-f(y)|^2 d\mu(x) d\mu(y) \le C \frac{1}{s^{d_W}} \int_X \int_X |f(x)-f(y)|^2 p_{s^{d_W}}(x,y) d\mu(x) d\mu(y).$$

This yields

$$\sup_{r>0} \frac{1}{r^{d_W}} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^2}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \le C \sup_{t>0} \frac{1}{t} \int_X \int_X |f(x)-f(y)|^2 p_t(x,y) d\mu(x) d\mu(y)$$

One now has

$$\begin{aligned}
&\int_X \int_X |f(x)-f(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \\
&= \int_X \int_X (f(x)^2 - 2f(x)f(y) + f(y)^2) p_t(x,y) d\mu(x) d\mu(y) \\
&= \int_X (P_t f^2)(y) d\mu(y) - 2 \int_X f(x) P_t f(x) d\mu(x) + \int_X (P_t f^2)(x) d\mu(x) \\
&= 2\langle f, f - P_t f \rangle_{L^2}
\end{aligned}$$

Since $\frac{1}{t}\langle f, f - P_t f \rangle_{L^2}$ is decreasing and converges to $\mathcal{E}(f)$ when $t \to 0$, one concludes

$$\sup_{r>0} \frac{1}{r^{d_W}} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^2}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x) \le C\mathcal{E}(f).$$

□

Our theorem follows therefore from the following lemma:

Lemma 3.3.3. *For $f \in \mathcal{F}$*

$$\mathcal{E}(f) \leq C \liminf_{r \to 0^+} \frac{1}{r^{d_W}} \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^2}{\mu(B(x,r))} \, d\mu(y) \, d\mu(x)$$

Proof. We first note that from the previous proof

$$\mathcal{E}(f) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X p_t(x,y) |f(x) - f(y)|^2 d\mu(x) \, d\mu(y)$$

Write

$$\Psi(t) = \frac{1}{t} \int_X \int_X p_t(x,y) |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y)$$

and estimate as follows. Let $r = \delta t^{1/d_W}$. For $d(x,y) < \delta t^{1/d_W}$ the sub-Gaussian upper bound (3.0.1) implies $p_t(x,y) \leq C t^{-d_H/d_W}$, so that

$$\begin{aligned}
&\frac{1}{t} \int_X \int_{B(y,r)} p_t(x,y) |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) \\
\leq &\frac{C}{t^{1+d_H/d_W}} \int_X \int_{B(y,r)} |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) \\
\leq &\frac{C}{t^{1+d_H/d_W}} \int_X \int_{B(y,\delta t^{1/d_W})} |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) := \Phi(t).
\end{aligned}$$

For $d(x,y) > \delta t^{1/d_W}$ we instead use the sub-Gaussian bounds (3.0.1) to see there are $c, C > 1$ such that

$$p_t(x,y) \leq C \exp\Big(-\Big(\frac{c_4}{2}\Big) \Big(\frac{d(x,y)^{d_W}}{t}\Big)^{\frac{1}{d_W - 1}} \Big) p_{ct}(x,y) \leq C \exp\big(-c' \delta^{\frac{d_W}{d_W-1}}\big) p_{ct}(x,y)$$

and therefore

$$\begin{aligned}
\Psi(t) &\leq \Phi(t) + \frac{1}{t} \int_X \int_{X \setminus B(y,r)} p_t(x,y) |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) \\
&\leq \Phi(t) + \frac{C}{t} \exp\big(-c' \delta^{\frac{d_W}{d_W-1}}\big) \int_X \int_{X \setminus B(y,r)} p_{ct}(x,y) |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y) \\
&= \Phi(t) + A \Psi(ct) \qquad (3.3.1)
\end{aligned}$$

where A is a constant that can be made as small as we desire by making δ large enough. We fix δ so that $A < \frac{1}{2}$. By letting $t \to 0$, one gets

$$\mathcal{E}(f) \leq \liminf_{t \to 0^+} \frac{1}{t^{1+d_H/d_W}} \int_X \int_{B(y,\delta t^{1/d_W})} |f(x) - f(y)|^2 \, d\mu(x) \, d\mu(y)$$

□

For $\lambda > 0$, we define the space $KS^{\lambda,2}(X)$ as the collection of all functions $f \in L^2(X,\mu)$ for which

$$\|f\|^p_{KS^{\lambda,2}(X)} := \limsup_{r\to 0^+} \int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^2}{r^{2\lambda}\mu(B(x,r))} \, d\mu(y)\, d\mu(x) < +\infty.$$

The L^2–Korevaar-Schoen critical exponents of the space are then defined as:

$$\lambda_2^* = \inf\left\{\lambda > 0 \,:\, KS^{\lambda,2}(X) \text{ is trivial}\right\},$$

$$\lambda_2^\dagger = \inf\left\{\lambda > 0 \,:\, KS^{\lambda,2}(X) \text{ is dense in } L^2\right\},$$

where by trivial we mean that $KS^{\lambda,2}(X)$ contains only almost everywhere constant functions.

Corollary 3.3.4.

$$\lambda_2^* = \lambda_2^\dagger = \frac{d_W}{2}.$$

Proof. We know that $KS^{d_W/2,2}(X) = \mathcal{F}$ is dense in L^2 and from the previous theorem $f \in KS^{\lambda,2}(X)$ with $\lambda > \frac{d_W}{2}$ implies that $\mathcal{E}(f) = 0$ and thus f constant. □

3.4 Regular Dirichlet forms, Energy measures

In this section, let X be a locally compact and complete metric space equipped with a Radon measure μ supported on X. Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a Dirichlet form on X. We assume throughout the section that the heat semigroup P_t is stochastically complete but we do not necessarily assume the sub-Gaussian estimates. We denote by $C_c(X)$ the vector space of all continuous functions with compact support in X and $C_0(X)$ its closure with respect to the supremum norm. A core for $(X, \mu, \mathcal{E}, \mathcal{F})$ is a subset $\mathcal{C}$ of $C_c(X) \cap \mathcal{F}$ which is dense in $C_c(X)$ in the supremum norm and dense in $\mathcal{F}$ in the norm

$$\left(\|f\|^2_{L^2(X,\mu)} + \mathcal{E}(f,f)\right)^{1/2}.$$

Definition 3.4.1. *The Dirichlet form $\mathcal{E}$ is called regular if it admits a core.*

Lemma 3.4.2. *For $f, g \in \mathcal{F} \cap L^\infty(X,\mu)$, $fg \in \mathcal{F}$ and*

$$\mathcal{E}(fg)^{1/2} \le \|f\|_\infty \mathcal{E}(g)^{1/2} + \|g\|_\infty \mathcal{E}(f)^{1/2}$$

Theorem 3.4.3 (Energy measures)**.** *Assume that $\mathcal{E}$ is regular. For $f \in \mathcal{F} \cap L^\infty(X,\mu)$, there exists a unique Radon measure on X denoted $d\Gamma(f)$ so that for every $\phi \in \mathcal{F} \cap C_c(X)$,*

$$\begin{aligned}\int_X \phi \, d\Gamma(f) &= \frac{1}{2}[2\mathcal{E}(\phi f, f) - \mathcal{E}(\phi, f^2)] \\ &= \lim_{t\to 0} \frac{1}{2t} \int_X \int_X \phi(x)(f(x)-f(y))^2 p_t(x,dy) d\mu(x).\end{aligned}$$

Proof.

$$\int_X \int_X \phi(x)(f(x)-f(y))^2 p_t(x,dy)d\mu(x) = -\langle (I-P_t)f^2, \phi\rangle + 2\langle (I-P_t)f, f\phi\rangle$$

□

Lemma 3.4.4. *Let $f \in \mathcal{F}$. Then $f_n = \min(n, \max(-n, u)) \in \mathcal{F}$ and $\mathcal{E}(f - f_n) \to 0$.*

Lemma 3.4.5. *For $f, g \in \mathcal{F} \cap L^\infty(X, \mu)$ and $\phi \in \mathcal{F} \cap C_c(X)$,*

$$\left| \sqrt{\int_X \phi\, d\Gamma(f)} - \sqrt{\int_X \phi\, d\Gamma(g)} \right|^2 \le \int_X \phi\, d\Gamma(f-g) \le \|\phi\|_{L^\infty(X,\mu)} \mathcal{E}(f-g)$$

Thanks to the previous lemmas, by approximation, one can define $d\Gamma(f)$ for every $f \in \mathcal{F}$. For $f, g \in \mathcal{F}$, one can define $d\Gamma(f, g)$ by polarization.

Theorem 3.4.6 (Beurling-Deny)**.** *For $u, v \in \mathcal{F}$*

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v).$$

Example 3.4.7 (Uniformly elliptic divergence form diffusion operators)**.** *On $\mathbb{R}^n$, we consider the divergence form operator*

$$Lf = -\mathbf{div}(\sigma \nabla f),$$

where σ is a smooth field of positive and symmetric matrices that satisfies

$$a\|x\|^2 \le \langle x, \sigma(y)x\rangle \le b\|x\|^2, \quad x, y \in \mathbb{R}^n,$$

for some constant $0 < a \le b$. Consider the Dirichlet form

$$\mathcal{E}(f) = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \sigma_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx, \quad f \in W^{1,2}(\mathbb{R}^n).$$

Then $\mathcal{E}$ is regular and for $f, g \in W^{1,2}(\mathbb{R}^n)$,

$$d\Gamma(f, g) = \langle \sigma \nabla f, \sigma \nabla g\rangle dx.$$

Example 3.4.8 (Riemannian manifolds)**.** *Let $(\mathbb{M}, g)$ be a complete n-dimensional Riemannian manifold with Riemannian volume measure μ. We consider the standard Dirichlet form $\mathcal{E}$ on $\mathbb{M}$, which is obtained by closing the bilinear form*

$$\mathcal{E}(f, g) = \int_{\mathbb{M}} \langle \nabla f, \nabla g\rangle d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

Then $\mathcal{E}$ is regular and for $f, g \in \mathcal{F}$,

$$d\Gamma(f, g) = \langle \nabla f,\ \nabla g\rangle dx.$$

Example 3.4.9 (Carnot groups)**.** *Let* $\mathbb{G}$ *be a Carnot group with sub-Laplacian*

$$L = \sum_{i=1}^{d} V_i^2$$

and Dirichlet form

$$\mathcal{E}(f) = \int_{\mathbb{G}} \sum_{i=1} (V_i f)^2 d\mu.$$

Then $\mathcal{E}$ *is regular and for* $f, g \in \mathcal{F}$*,*

$$d\Gamma(f,g) = \sum_{i=1}^{d} (V_i f)(V_i g) dx.$$

Example 3.4.10. *Consider on the infinite Sierpinski gasket the Dirichlet form*

$$\mathcal{E}(f) := \lim_{n \to +\infty} \left(\frac{5}{3} \right)^n \mathcal{E}_n(f).$$

Then $\mathcal{E}$ *is regular, but unless* f *is constant, for* $f \in \mathcal{F}$*,* $d\Gamma(f)$ *is singular with respect to the Hausdorff measure* μ*.*

3.5 Energy measure estimates

We now come back in this section to the framework outlined at the beginning of the chapter, i.e. we assume sub-Gaussian heat kernel estimates.
We first prove that the form $\mathcal{E}$ has to be regular. This will follow from a property of the semigroup P_t which is called the Feller property.

Definition 3.5.1. *The heat semigroup* $(P_t)_{t \ge 0}$ *is called a Feller semigroup if:*

1. *For every* $t > 0$*,* $P_t(C_0(X)) \subset C_0(X)$*;*
2. *For every* $f \in C_0(X)$*,*
$$\lim_{t \to 0} \|P_t f - f\|_{L^\infty(X,\mu)} = 0.$$

Lemma 3.5.2. *The semigroup* $(P_t)_{t \ge 0}$ *is a Feller semigroup.*

Proof. 1. Let $f \in C_0(X)$, $t > 0$ and $\varepsilon > 0$. Since $f \in C_0(X)$, there exists a compact set $K \subset X$ so that for every $y \in X \setminus K$, $|f(y)| \le \varepsilon$. One has then

$$\begin{aligned}
|P_t f(x)| &= \left| \int_X p_t(x,y) f(y) d\mu(y) \right| \\
&\le \int_K p_t(x,y) |f(y)| d\mu(y) + \varepsilon \\
&\le c_3 t^{-d_H/d_W} \exp\left(-c_4 \left(\frac{d(x,K)^{d_W}}{t} \right)^{\frac{1}{d_W - 1}} \right) \int_K |f(y)| d\mu(y) + \varepsilon
\end{aligned}$$

Thus, $P_t f$ vanishes at ∞. The fact that $P_t f$ is continuous easily follows from the joint continuity of the heat kernel and the proof is let to the reader.

2. Let $f \in C_0(X)$ and $\varepsilon > 0$. Since $f \in C_0(X)$, it is uniformly continuous. Thus there exists $r > 0$ so that $d(x,y) \le r$ implies $|f(x) - f(y)| \le \varepsilon$.

   From Lemma 3.2.3, one has then

$$\begin{aligned}
|P_t f(x) - f(x)| &= \left| \int_X p_t(x,y) f(y) d\mu(y) - f(x) \right| \\
&= \left| \int_X p_t(x,y)(f(y) - f(x)) d\mu(y) \right| \\
&\le \int_X p_t(x,y) |f(y) - f(x)| d\mu(y) \\
&\le \int_{X \setminus B(x,r)} p_t(x,y) |f(y) - f(x)| d\mu(y) + \int_{B(x,r)} p_t(x,y) |f(y) - f(x)| d\mu(y) \\
&\le 2C \exp\left(-\frac{1}{C} \left(\frac{r^{d_W}}{t} \right)^{\frac{1}{d_W - 1}} \right) \|f\|_{L^\infty(X,\mu)} + \varepsilon.
\end{aligned}$$

□

As an immediate corollary of the Feller property, one deduces regularity.

Theorem 3.5.3. *$\mathcal{E}$ is regular*

Proof. If one defines

$$\mathcal{C} = \{P_t f, f \in C_0(X), t > 0\},$$

then it is a subset of $C_0(X) \cap \mathcal{F}$ which is dense in $C_0(X)$ in the supremum norm and dense in $C_0(X) \cap \mathcal{F}$ in the $\mathcal{E}_1$-norm

$$\left(\|f\|^2_{L^2(X,\mu)} + \mathcal{E}(f,f) \right)^{1/2}.$$

It remains to prove that $C_0(X) \cap \mathcal{F}$ is dense in $\mathcal{F}$ in the $\mathcal{E}_1$-norm, which follows from the fact that the heat kernel estimates imply $P_t(L^2(X,\mu)) \subset C_0(X) \cap \mathcal{F}$ for $t > 0$ (Exercise !). □

One now turns to a metric estimate of the energy measures.

Theorem 3.5.4. *There exist constants $c, C > 0$ such that for every $f \in \mathcal{F} \cap L^\infty(X,\mu)$ and associated energy measure $d\Gamma(f)$ we have for every $g \in \mathcal{F} \cap C_0(X)$, $g \ge 0$,*

$$\begin{aligned}
& c \limsup_{r \to 0^+} \frac{1}{r^{d_W}} \int_X g(x) \left(\int_{B(x,r)} \frac{|f(y) - f(x)|^2}{\mu(B(x,r))} d\mu(y) \right) d\mu(x) \\
\le & \int_X g \, d\Gamma(f) \\
\le & C \liminf_{r \to 0^+} \frac{1}{r^{d_W}} \int_X g(x) \left(\int_{B(x,r)} \frac{|f(y) - f(x)|^2}{\mu(B(x,r))} d\mu(y) \right) d\mu(x).
\end{aligned}$$

Proof. We first prove the lower bound. For $s, t > 0$ and $\alpha > 0$,

$$\begin{aligned}
&\int_X \int_X |f(x) - f(y)|^2 p_t(x, y) d\mu(x) g(y)\, d\mu(y) \\
&\quad \geq \int_X \int_{B(y,s)} |f(x) - f(y)|^2 p_t(x, y) d\mu(x) g(y)\, d\mu(y) \\
&\quad \geq c_1 t^{-d_H/d_W} \int_X \int_{B(y,s)} |f(x) - f(y)|^2 \exp\left(-c_2 \left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W - 1}}\right) d\mu(x) g(y)\, d\mu(y) \\
&\quad \geq c_1 t^{-d_H/d_W} \exp\left(-c_2 \left(\frac{s^{d_W}}{t}\right)^{\frac{1}{d_W - 1}}\right) \int_X \int_{B(y,s)} |f(x) - f(y)|^2 d\mu(x) g(y)\, d\mu(y).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\frac{1}{s^{d_H + d_W}} \int_X \int_{B(y,s)} |f(x) - f(y)|^2 d\mu(x) g(y)\, d\mu(y) \\
\leq & C \frac{t^{d_H/d_W}}{s^{d_H + d_W}} \exp\left(c_2 \left(\frac{s^{d_W}}{t}\right)^{\frac{1}{d_W - 1}}\right) \int_X \int_X |f(x) - f(y)|^2 p_t(x, y) d\mu(x) g(y)\, d\mu(y)
\end{aligned}$$

Choosing $t = s^{d_W}$, one deduces

$$\frac{1}{t^{1 + d_H/d_W}} \int_X \int_{B(y, t^{1/d_W})} |f(x) - f(y)|^2 d\mu(x) d\mu(y) \leq C \frac{1}{t} \int_X \int_X |f(x) - f(y)|^2 p_t(x, y) d\mu(x) d\mu(y).$$

By letting $t \to 0$ we get the lower bound

$$c \limsup_{r \to 0^+} \frac{1}{r^{d_W}} \int_X g(x) \left(\int_{B(x,r)} \frac{|f(y) - f(x)|^2}{\mu(B(x,r))} d\mu(y) \right) d\mu(x) \leq \int_X g d\Gamma(f).$$

The proof of the upper bound follows closely the proof of Lemma 3.3.3, so is omitted for conciseness. □

3.6 Strongly local Dirichlet forms

In this section, we still consider a Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$ on a metric measure space (X, d, μ) whose heat kernel satisfies the sub-Gaussian estimates (3.0.1).

Definition 3.6.1. *The Dirichlet form is called* strongly local *if for any two functions $f, g \in \mathcal{F}$ with compact supports such that f is constant in a neighborhood of the support of g, we have $\mathcal{E}(f, g) = 0$.*

Theorem 3.6.2. *The Dirichlet form $\mathcal{E}$ is strongly local.*

Proof. Let K be the support of g and assume that f is constant on an open set $\mathcal{O}$ that contains K. One has,

$$\begin{aligned}
&\int_X \int_X (f(x)-f(y))(g(x)-g(y))p_t(x,y)d\mu(x)d\mu(y) \\
=&\int_{X\setminus\mathcal{O}} \int_K (f(x)-f(y))(g(x)-g(y))p_t(x,y)d\mu(x)d\mu(y) \\
&+\int_K \int_{X\setminus\mathcal{O}} (f(x)-f(y))(g(x)-g(y))p_t(x,y)d\mu(x)d\mu(y)
\end{aligned}$$

One can estimate

$$\begin{aligned}
&\int_K \int_{X\setminus\mathcal{O}} |f(x)-f(y)||g(x)-g(y)|p_t(x,y)d\mu(x)d\mu(y) \\
\le& c_3 t^{-d_H/d_W} \exp\left(-c_4\Big(\frac{d(K,X\setminus\mathcal{O})^{d_W}}{t}\Big)^{\frac{1}{d_W-1}}\right)\int_K \int_{X\setminus\mathcal{O}} |f(x)-f(y)||g(x)-g(y)|d\mu(x)d\mu(y).
\end{aligned}$$

Thus,

$$\mathcal{E}(f,g)=\lim_{t\to 0}\frac{1}{2t}\int_X\int_X (f(x)-f(y))(g(x)-g(y))p_t(x,y)d\mu(x)d\mu(y)=0.$$

□

Definition 3.6.3. *The energy measures $\Gamma(u,v)$ inherit a strong locality property from $\mathcal{E}$, namely that $\mathbf{1}_U d\Gamma(u,v)=0$ for any open subset $U\subset X$ and $u,v\in\mathcal{F}$ such that u is a constant on U. One can then extend Γ to $\mathcal{F}_{\mathrm{loc}}(X)$ defined as*

$$\mathcal{F}_{\mathrm{loc}}(X)=\{u\in L^2_{\mathrm{loc}}(X):\forall \text{ compact } K\subset X, \exists v\in\mathcal{F} \text{ such that } u=v|_K \text{ a.e.}\}.$$

We will still denote this extension by Γ. We collect some properties of this extension.

- *Strong locality.* For all $u,v\in\mathcal{F}_{\mathrm{loc}}(X)$ and all open subset $U\subset X$ on which u is a constant
$$\mathbf{1}_U d\Gamma(u,v)=0.$$
- *Leibniz and chain rules.* For all $u\in\mathcal{F}_{\mathrm{loc}}(X), v\in\mathcal{F}_{\mathrm{loc}}(X)\cap L^\infty_{\mathrm{loc}}(X)$, $w\in\mathcal{F}_{\mathrm{loc}}(X)$ and $\eta\in C^1(\mathbb{R})$,
$$\begin{aligned}
d\Gamma(uv,w)&=ud\Gamma(v,w)+vd\Gamma(u,w),\\
d\Gamma(\eta(u),v)&=\eta'(u)d\Gamma(u,v).
\end{aligned}$$

Chapter 4

Strictly local Dirichlet spaces

Throughout the chapter, let X be a locally compact and complete metric space equipped with a Radon measure μ supported on X. Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a Dirichlet form on X. We assume throughout that the heat semigroup P_t is stochastically complete and that the form $\mathcal{E}$ is strongly local and regular.

4.1 Intrinsic metric

We denote by $C_c(X)$ the vector space of all continuous functions with compact support in X and $C_0(X)$ its closure with respect to the supremum norm.
With respect to $\mathcal{E}$ we can define the following *intrinsic metric* $d_{\mathcal{E}}$ on X by

$$d_{\mathcal{E}}(x,y) = \sup\{u(x) - u(y) \,:\, u \in \mathcal{F} \cap C_0(X) \text{ and } d\Gamma(u,u) \le d\mu\}. \tag{4.1.1}$$

Here the condition $d\Gamma(u,u) \le d\mu$ means that $\Gamma(u,u)$ is absolutely continuous with respect to μ with Radon-Nikodym derivative bounded by 1.
The term "intrinsic metric" is potentially misleading because in general there is no reason why $d_{\mathcal{E}}$ is a metric on X (it could be infinite for a given pair of points x, y or zero for some distinct pair of points).

Definition 4.1.1. *A strongly local regular Dirichlet space is called strictly local if $d_{\mathcal{E}}$ is a metric on X and the topology induced by $d_{\mathcal{E}}$ coincides with the topology on X.*

Example 4.1.2 (Uniformly elliptic divergence form diffusion operators)**.** *On $\mathbb{R}^n$, we consider the divergence form operator*

$$Lf = -\mathbf{div}(\sigma \nabla f),$$

where σ is a smooth field of positive and symmetric matrices that satisfies

$$a\|x\|^2 \le \langle x, \sigma(y)x \rangle \le b\|x\|^2, \quad x, y \in \mathbb{R}^n,$$

for some constant $0 < a \le b$. Consider the Dirichlet form

$$\mathcal{E}(f) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx, \quad f \in W^{1,2}(\mathbb{R}^n).$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x,y) \simeq \|x-y\|.$$

Example 4.1.3 (Riemannian manifolds)**.** *Let $(\mathbb{M}, g)$ be a complete n-dimensional Riemannian manifold with Riemannian volume measure μ. We consider the standard Dirichlet form $\mathcal{E}$ on $\mathbb{M}$, which is obtained by closing the bilinear form*

$$\mathcal{E}(f,g) = \int_{\mathbb{M}} \langle \nabla f, \nabla g \rangle d\mu, \quad f,g \in C_0^\infty(\mathbb{M}).$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x,y) = d_g(x,y).$$

Example 4.1.4 (Carnot groups)**.** *Let $\mathbb{G}$ be a Carnot group with sub-Laplacian*

$$L = \sum_{i=1}^{d} V_i^2$$

and Dirichlet form

$$\mathcal{E}(f) = \int_{\mathbb{G}} \sum_{i=1} (V_i f)^2 d\mu.$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x,y) = d_{CC}(x,y)$$

where d_{CC} is the so-called Carnot-Carathéodory distance which is defined as follows.
An absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is said to be subunit for the operator L if for every smooth function $f : \mathbb{G} \to \mathbb{R}$ we have $\left|\frac{d}{dt} f(\gamma(t))\right| \le \sqrt{(\Gamma f)(\gamma(t))}$. We then define the subunit length of γ as $\ell_s(\gamma) = T$.
Given $x, y \in \mathbb{G}$, we indicate with

$$S(x,y) = \{\gamma : [0,T] \to \mathbb{G} \mid \gamma \text{ is subunit for } \Gamma, \gamma(0) = x,\ \gamma(T) = y\}.$$

It is a consequence of the Chow-Rashevskii theorem that

$$S(x,y) \neq \emptyset, \quad \text{for every } x, y \in \mathbb{G}.$$

One defines then

$$d_{CC}(x,y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x,y)\}, \tag{4.1.2}$$

Example 4.1.5. *Consider on the infinite Sierpinski gasket the Dirichlet form*

$$\mathcal{E}(f) := \lim_{n \to +\infty} \left(\frac{5}{3}\right)^n \mathcal{E}_n(f).$$

Then $\mathcal{E}$ is regular, but unless f is constant, for $f \in \mathcal{F}$, $d\Gamma(f)$ is singular with respect to the Hausdorff measure μ. As a consequence $\mathcal{E}$ is not strictly local.

4.2 Volume doubling property and Poincaré inequality

Definition 4.2.1. *We say that the metric measure space $(X, d_{\mathcal{E}}, \mu)$ satisfies the volume doubling property if there exists a constant $C > 0$ such that for every $x \in X$ and $r > 0$,*

$$\mu(B(x, 2r)) \leq C\,\mu(B(x, r)).$$

It is easily seen that it follows from the doubling property of μ that there is a constant $0 < Q < \infty$ and $C \geq 1$ such that whenever $0 < r \leq R$ and $x \in X$, we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C \left(\frac{R}{r}\right)^Q. \tag{4.2.1}$$

Definition 4.2.2. *We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ supports the 2-Poincaré inequality if there are constants $C > 0$ and $\lambda \geq 1$ such that whenever B is a ball in X (with respect to the metric $d_{\mathcal{E}}$) and $u \in \mathcal{F}$, we have*

$$\int_B |u - u_B|^2\, d\mu \leq C\, \mathrm{rad}\,(B)^2 \int_{\lambda B} d\Gamma(u, u).$$

A theorem due to Grigorian, Saloff-Coste and Sturm is the following:

Theorem 4.2.3. *Let $(X, d = d_{\mathcal{E}}, \mathcal{E}, \mu, \mathcal{F})$ be a strictly local Dirichlet space. The following are equivalent:*

1. *$(X, \mu, \mathcal{E}, \mathcal{F})$ supports the 2-Poincaré inequality and $(X, d_{\mathcal{E}}, \mu)$ satisfies the volume doubling property;*
2. *The heat semigroup P_t admits a jointly continuous heat kernel function $p_t(x, y)$ on $[0, \infty) \times X \times X$ for which there are constants $c_1, c_2, C > 0$ such that whenever $t > 0$ and $x, y \in X$,*
$$\frac{1}{C}\,\frac{e^{-c_1 d(x,y)^2/t}}{\mu(B(x,\sqrt{t}))} \leq p_t(x, y) \leq C\,\frac{e^{-c_2 d(x,y)^2/t}}{\mu(B(x,\sqrt{t}))}. \tag{4.2.2}$$

The proof is divided into several lemmas.

Lemma 4.2.4. *Assume* (4.2.2). *Then $(X, d_{\mathcal{E}}, \mu)$ satisfies the volume doubling property.*

Proof.

$$\begin{aligned}
&\int_X p_t(x,y)d\mu(y)\\
\geq &\int_{B(x,R)} p_t(x,y)d\mu(y)\\
\geq &\int_{B(x,R)} \frac{1}{C}\,\frac{e^{-c_1 d(x,y)^2/t}}{\mu(B(x,\sqrt{t}))}d\mu(y)\\
\geq &\frac{1}{C}\,\frac{e^{-c_1 R^2/t}}{\mu(B(x,\sqrt{t}))}\mu(B(x,R)).
\end{aligned}$$

Since $\int_X p_t(x,y)d\mu(y) = 1$, one deduces

$$\mu(B(x,R)) \le Ce^{c_1 R^2/t}\mu(B(x,\sqrt{t})).$$

Choosing $t = R^2/4$ yields the expected result. □

Lemma 4.2.5. *Assume* (4.2.2). *Then* $(X, \mu, \mathcal{E}, \mathcal{F})$ *supports the* 2*-Poincaré inequality.*

Proof. The proof uses the Neumann heat kernel and we skip some of the details. Let $x_0 \in X$ and $r > 0$. We denote $\Omega = B(x_0, r)$. Let $\mathcal{D}_\Omega$ be the set of functions $f \in \mathcal{F}_{loc}$ such that for every $g \in \mathcal{F}_{loc}$,

$$\int_\Omega gLf d\mu = -\int_\Omega d\Gamma(f,g).$$

It is easy to see that L is essentially self-adjoint on $\mathcal{D}_\Omega$. Its Friedrichs extension, still denoted L, is called the Neumann Laplacian on Ω and the semigroup it generates, the Neumann semigroup. Denote this semigroup by P_t^N. By using the global lower bound

$$p(x,y,t) \ge \frac{C}{\mu(B(x,\sqrt{t}))}\exp\left(-C\frac{d(x,y)^2}{t}\right),$$

for the heat kernel, it is possible to prove a lower bound for the Neumann heat kernel on the ball $B(x_0, r)$: For $x, y \in B(x_0.r/2)$,

$$p^N(x,y,r^2) \ge \frac{C}{\mu(B(x_0,r/2))}.$$

We have for $f \in \mathcal{D}_\Omega$

$$P^N_{r^2/2}(f^2) - (P^N_{r^2/2}f)^2 = \int_0^{r^2/2} \frac{d}{dt}P_t^N((P^N_{r^2/2-t}f)^2)dt.$$

By integrating over Ω, we find then,

$$\begin{aligned}\int_\Omega P^N_{r^2/2}(f^2) - (P^N_{r^2/2}f)^2 d\mu &= -\int_0^{r^2/2}\int_\Omega \frac{d}{dt}(P_t^N f)^2 d\mu dt \\ &= 2\int_0^{r^2/2}\int_\Omega d\Gamma(P_t^N f, P_t^N f)dt \\ &\le r^2 \int_\Omega d\Gamma(f).\end{aligned}$$

But on the other hand, we have

$$\begin{aligned}\int_\Omega P^N_{r^2/2}(f^2) - (P^N_{r^2/2}f)^2 d\mu &= \frac{1}{2}\int_\Omega\int_\Omega p^N_{r^2}(x,y)(f(x)-f(y))^2 d\mu(x)\, d\mu(y) \\ &\ge \frac{1}{2}\int_{\Omega/2}\int_{\Omega/2} p^N_{r^2}(x,y)(f(x)-f(y))^2 d\mu(x)\, d\mu(y) \\ &\ge \frac{C}{\mu(\Omega/2)}\int_{\Omega/2}\int_{\Omega/2}(f(x)-f(y))^2 d\mu(x)\, d\mu(y)\end{aligned}$$

which gives

$$\int_{\Omega} P^N_{r^2/2}(f^2) - (P^N_{r^2/} f)^2 d\mu \geq C \int_{\Omega/2} \left(f(x) - \frac{1}{\mu(\Omega/2)} \int_{\Omega/2} f d\mu \right)^2 d\mu(x)$$

The proof is complete.

□