A short introduction to Dirichlet Spaces

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Abstract

We provide an overview of the fundamentals in the theory of Dirichlet forms. Dirichlet forms theory allows us to define Laplacians, PDEs and boundary conditions in very general frameworks which do not require any kind of smooth structures including metric spaces like fractals. In this mini-course, we will cover the following topics:

- Contraction semigroups, quadratic forms and generators in Hilbert spaces;
- Dirichlet forms;
- Examples of Dirichlet spaces: Riemannian manifolds, Fractals, Metric spaces;
- The Gagliardo-Nirenberg interpolation theory in Dirichlet spaces.
- Further topics

Contents

Chapter 1

Semigroups, generators and quadratic forms on Hilbert spaces

1.1 Preliminaries: Self-adjoint operators

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space with norm $\|f\|^2 = \langle f, f \rangle$ and let A be a $\mathcal{H}$-valued densely defined operator on a domain $\mathcal{D}(A)$. We recall the following basic definitions.

- The operator A is said to be closed if $x_n \to x$ in $\mathcal{H}$ and $Ax_n \to y$ in $\mathcal{H}$ imply that $y = Ax$.
- The operator A is said to be symmetric if for $f, g \in \mathcal{D}(A)$,

$$\langle f, Ag \rangle = \langle Af, g \rangle.$$

- The operator A is said to be non negative symmetric operator if it is symmetric and if for $f \in \mathcal{D}(A)$,

$$\langle f, Af \rangle \geq 0.$$

  It is said to be non positive, if for $f \in \mathcal{D}(A)$,

$$\langle f, Af \rangle \leq 0.$$

- The adjoint A^* of A is an operator defined on the domain

$$\mathcal{D}(A^*) = \{ f \in \mathcal{H} : \exists\ c(f) \geq 0, \forall\ g \in \mathcal{D}(A), |\langle f, Ag \rangle| \leq c(f) \|g\| \}.$$

  Since for $f \in \mathcal{D}(A^*)$, the map $g \to \langle f, Ag \rangle$ is bounded on $\mathcal{D}(A)$, it extends thanks to Hahn-Banach theorem to $\mathcal{H}$. The Riesz representation theorem allows then to define A^* by the formula

$$\langle A^* f, g \rangle = \langle f, Ag \rangle$$

  where $g \in \mathcal{D}(A), f \in \mathcal{D}(A^*)$. Since $\mathcal{D}(A)$ is dense, A^* is uniquely defined.

- The operator A is said to be self-adjoint if it is symmetric and if $\mathcal{D}(A^*) = \mathcal{D}(A)$.

Let us observe that, in general, the adjoint A^* is not necessarily densely defined, however it is readily checked that if A is a symmetric operator then, from Cauchy-Schwarz inequality, $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. Thus, if A is symmetric, then A^* is densely defined. The following result is often useful.

Lemma 1.1.1. *Let $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ be an injective densely defined self-adjoint operator. Let us denote by $\mathcal{R}(A)$ the range of A. The inverse operator*

$$A^{-1} : \mathcal{R}(A) \to \mathcal{H}$$

is a densely defined self-adjoint operator.

A major result in functional analysis is the spectral theorem.

Theorem 1.1.2 (Spectral theorem)**.** *Let A be a non negative self-adjoint operator on $\mathcal{H}$. There is a measure space (Ω, ν), a unitary map $U : L^2(\Omega, \nu) \to \mathcal{H}$ and a non negative real valued measurable function λ on Ω such that*

$$U^{-1}AUf(x) = \lambda(x)f(x),$$

for $x \in \Omega$, $Uf \in \mathcal{D}(A)$. Moreover, given $f \in L^2(\Omega, \nu)$, Uf belongs to $\mathcal{D}(A)$ if only if $\int_\Omega \lambda^2 f^2 d\nu < +\infty$.

Definition 1.1.3. *(Functional calculus) Let A be a non negative self-adjoint operator on $\mathcal{H}$. Let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a Borel function. With the notations of the spectral theorem, one defines the operator $g(A)$ by the requirement*

$$U^{-1}g(A)Uf(x) = g(\lambda(x))f(x),$$

with $\mathcal{D}(g(A)) = \{Uf, (g \circ \lambda)f \in L^2(\Omega, \nu)\}$.

Exercise 1.1.4. *Show that if A is a non negative self-adjoint operator on $\mathcal{H}$ and g is a bounded Borel function, then $g(A)$ is a bounded operator on $\mathcal{H}$.*

1.2 Semigroups and generators

Definition 1.2.1. *A strongly continuous self-adjoint contraction semigroup is a family of self-adjoint operators $(P_t)_{t\geq 0} : \mathcal{H} \to \mathcal{H}$ such that:*

1. *For $s, t \geq 0$, $P_t \circ P_s = P_{s+t}$ (semigroup property);*

2. *For every $f \in \mathcal{H}$, $\lim_{t\to 0} P_t f = f$ (strong continuity);*

3. *For every $f \in \mathcal{H}$ and $t \geq 0$, $\|P_t f\| \leq \|f\|$ (contraction property).*

Theorem 1.2.2. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$. There exists a self-adjoint, non-positive and densely defined operator*

$$A : \mathcal{D}(A) \to \mathcal{H}$$

where

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H} : \quad \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists} \right\},$$

such that for $f \in \mathcal{D}(A)$,

$$\lim_{t\to 0} \left\| \frac{P_t f - f}{t} - Af \right\| = 0.$$

The operator A is called the generator of the semigroup $(P_t)_{t\geq 0}$. We also say that A generates $(P_t)_{t\geq 0}$. Conversely, if A is a densely defined non-positive self-adjoint operator on $\mathcal{H}$, then it is the generator of the strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$ defined as $P_t = e^{tA}$.

Proof. We first notice that the strong continuity assumption together with the semigroup property imply that for every $f \in \mathcal{H}$, $s \to P_s f$ is continuous. Let us then consider the following bounded operators on $\mathcal{H}$:

$$A_t = \frac{1}{t} \int_0^t P_s ds.$$

For $f \in \mathcal{H}$ and $h > 0$, we have

$$\begin{aligned}
\frac{1}{t}\left(P_t A_h f - A_h f\right) &= \frac{1}{ht} \int_0^h (P_{s+t} f - P_s f) ds \\
&= \frac{1}{ht} \left[\int_t^{h+t} P_s f ds - \int_0^h P_s f ds \right] \\
&= \frac{1}{ht} \left[\int_h^{h+t} P_s f ds + \int_t^h P_s f ds - \int_0^h P_s f ds \right] \\
&= \frac{1}{ht} \int_0^t (P_{s+h} f - P_s f) ds.
\end{aligned}$$

Therefore, we obtain

$$\lim_{t\to 0} \frac{1}{t}\left(P_t A_h f - A_h f\right) = \frac{1}{h}\left(P_h f - f\right).$$

This implies,

$$\{A_h f : x \in \mathcal{H}, h > 0\} \subset \left\{ f \in \mathcal{H} : \quad \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists} \right\}$$

Since $\lim_{h\to 0} A_h f = f$, we deduce that

$$\left\{ f \in \mathcal{H} : \quad \lim_{t\to 0} \frac{P_t f - f}{t} \text{ exists} \right\}$$

is dense in $\mathcal{H}$. We can then consider

$$Af := \lim_{t \to 0} \frac{P_t f - f}{t},$$

which is of course defined on the domain

$$\mathcal{D}(A) = \left\{ f \in \mathcal{H} : \quad \lim_{t \to 0} \frac{P_t f - f}{t} \text{ exists} \right\}.$$

The operator A is closed, indeed if $f_n \to f$ and $Af_n \to g$ then, using similar computations as before,

$$\begin{aligned} A_h g = \frac{1}{h} \int_0^h P_s g ds &= \lim_{n \to +\infty} \frac{1}{h} \int_0^h P_s A f_n ds \\ &= \lim_{n \to +\infty} \lim_{t \to 0} \frac{1}{ht} \int_0^h P_{s+t} f_n - P_s f_n ds \\ &= \lim_{n \to +\infty} \lim_{t \to 0} \frac{1}{ht} \int_0^t P_{s+h} f_n - P_s f_n ds \\ &= \lim_{n \to +\infty} \frac{1}{h} (P_h f_n - f_n) = \frac{1}{h} (P_h f - f) \end{aligned}$$

Taking then the limit as $h \to 0$ yields $y = Ax$. We now prove that A is a non positive self-adjoint operator. First, one has for every $f \in \mathcal{H}$

$$\begin{aligned} \langle Af, f \rangle &= \left\langle \lim_{t \to 0} \frac{P_t f - f}{t}, x \right\rangle \\ &= \lim_{t \to 0} \frac{\langle P_t f, f \rangle - \|f\|^2}{t} \\ &= \lim_{t \to 0} \frac{\|P_{t/2} f\|^2 - \|f\|^2}{t} \leq 0 \end{aligned}$$

From its definition, it is plain that A is symmetric but proving self-adjointness is a little more involved. Let $\lambda > 0$. We will to prove that $\lambda \mathrm{Id} - A$ is a bijective operator $\mathcal{D}(A) \to \mathcal{H}$ whose inverse is self-adjoint and conclude with the lemma 1.1.1.
The formal Laplace transform formula

$$\int_0^{+\infty} e^{-\lambda t} e^{tA} dt = (\lambda \mathrm{Id} - A)^{-1},$$

suggests that the operator

$$\mathbf{R}_\lambda = \int_0^{+\infty} e^{-\lambda t} P_t dt$$

is the inverse of $\lambda \mathrm{Id} - A$. We show this is indeed the case. First, let us observe that $\mathbf{R}_\lambda$ is well-defined as a Riemann integral since $t \to P_t$ is continuous and $\|P_t\| \leq 1$. We now show that for $f \in \mathcal{H}$, $\mathbf{R}_\lambda x \in \mathcal{D}(A)$. For $h > 0$,

$$\begin{aligned}
\frac{P_h - \mathrm{Id}}{h}\mathbf{R}_\lambda f &= \int_0^{+\infty} e^{-\lambda t}\frac{P_h - \mathrm{Id}}{h}P_t f dt \\
&= \int_0^{+\infty} e^{-\lambda t}\frac{P_{h+t} - P_t}{h} f dt \\
&= e^{\lambda h}\int_h^{+\infty} e^{-\lambda s}\frac{P_s - P_{s-h}}{h} f ds \\
&= \frac{e^{\lambda h}}{h}\left(\mathbf{R}_\lambda f - \int_0^h e^{-\lambda s}P_s f ds - \int_h^{+\infty} e^{-\lambda s}P_{s-h} f ds\right) \\
&= \frac{e^{\lambda h} - 1}{h}\mathbf{R}_\lambda f - \frac{e^{\lambda h}}{h}\int_0^h e^{-\lambda s}P_s f ds
\end{aligned}$$

By letting $h \to 0$, we deduce that $\mathbf{R}_\lambda f \in \mathcal{D}(A)$ and moreover

$$A\mathbf{R}_\lambda f = \lambda \mathbf{R}_\lambda f - f.$$

Therefore we proved

$$(\lambda \mathrm{Id} - A)\mathbf{R}_\lambda = \mathrm{Id}.$$

Furthermore, it is readily checked that, since A is closed, for $f \in \mathcal{D}(A)$,

$$A\mathbf{R}_\lambda f = A\int_0^{+\infty} e^{-\lambda t}P_t f dt = \int_0^{+\infty} e^{-\lambda t}AP_t f dt = \int_0^{+\infty} e^{-\lambda t}P_t A f dt = \mathbf{R}_\lambda A f.$$

We therefore conclude

$$(\lambda \mathrm{Id} - A)\mathbf{R}_\lambda = \mathbf{R}_\lambda(\lambda \mathrm{Id} - A) = \mathrm{Id}.$$

Thus,

$$\mathbf{R}_\lambda = (\lambda \mathrm{Id} - A)^{-1},$$

The operator $\int_0^{+\infty} e^{-\lambda t}P_t dt$ is seen to be self-adjoint (it is symmetric and bounded), thus $(\lambda \mathrm{Id} - A)^{-1}$ is also self-adjoint. From the previous lemma 1.1.1, we deduce that $\lambda \mathrm{Id} - A$ is self-adjoint, from which we conclude that A is self-adjoint (exercise !).
Conversely, let A be a densely defined non positive self-adjoint operator on $\mathcal{H}$ and define $P_t = e^{tA}$. More precisely, from spectral theorem, there is a measure space (Ω, ν), a unitary map $U : L^2(\Omega, \nu) \to \mathcal{H}$ and a non negative real valued measurable function λ on Ω such that

$$U^{-1}AUf(x) = -\lambda(x)f(x),$$

for $x \in \Omega$, $Uf \in \mathcal{D}(A)$. We define then $P_t : \mathcal{H} \to \mathcal{H}$ such that

$$U^{-1}P_tUf(x) = e^{-t\lambda(x)}f(x),$$

and let as an exercise the proof that $(P_t)_{t\geq 0}$ is a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$ with generator A. □

1.3 Quadratic forms and generators

Definition 1.3.1. *A quadratic form $\mathcal{E}$ on $\mathcal{H}$ is a non-negative definite, symmetric bilinear form $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to \mathbb{R}$, where $\mathcal{D}(\mathcal{E})$ is a dense subspace of $\mathcal{H}$. A quadratic form $\mathcal{E}$ on $\mathcal{H}$ is said to be closed if $\mathcal{D}(\mathcal{E})$ equipped with the norm*

$$\|f\|^2_{\mathcal{D}(\mathcal{E})} = \|f\|^2 + \mathcal{E}(f,f)$$

is a Hilbert space. A quadratic form $\mathcal{E}$ on $\mathcal{H}$ is said to be closable it admits a closed extension, i.e. there exists a closed quadratic form $\overline{\mathcal{E}}$ such that $\mathcal{D}(\mathcal{E}) \subset \mathcal{D}(\overline{\mathcal{E}})$ and $\overline{\mathcal{E}}$ coincides with $\mathcal{E}$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$.

Lemma 1.3.2. *A quadratic form $\mathcal{E}$ is closable if and only if for any sequence f_n in $\mathcal{D}(\mathcal{E})$ such that $f_n \to 0$ in $\mathcal{H}$ and $\mathcal{E}(f_n - f_m, f_n - f_m) \to 0$ when $n, m \to +\infty$ one has $\mathcal{E}(f_n, f_n) \to 0$.*

Proof. On $\mathcal{D}(\mathcal{E})$, let us consider the following norm

$$\|f\|^2_{\mathcal{E}} = \|f\|^2 + \mathcal{E}(f,f).$$

By completing $\mathcal{D}(\mathcal{E})$ with respect to this norm, we get an abstract Hilbert space $(\mathcal{H}_{\mathcal{E}}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$. Since for $f \in \mathcal{D}(\mathcal{E})$, $\|f\| \le \|f\|_{\mathcal{E}}$, the injection map $\iota : (\mathcal{D}(\mathcal{E}), \|\cdot\|_{\mathcal{E}}) \to (\mathcal{H}, \|\cdot\|)$ is continuous and it may therefore be extended into a continuous map $\bar{\iota} : (\mathcal{H}_{\mathcal{E}}, \|\cdot\|_{\mathcal{E}}) \to (\mathcal{H}, \|\cdot\|)$. Let us show that $\bar{\iota}$ is injective so that $\mathcal{H}_{\mathcal{E}}$ may be identified with a subspace of $\mathcal{H}$. So, let $f \in \mathcal{H}_{\mathcal{E}}$ such that $\bar{\iota}(f) = 0$. We can find a sequence $f_n \in \mathcal{D}(\mathcal{E})$, such that $\|f_n - f\|_{\mathcal{E}} \to 0$ and $\|f_n\| \to 0$. We have then

$$\begin{aligned}
\|f\|^2_{\mathcal{E}} &= \lim_{n\to+\infty} \langle f_n, f_n \rangle_{\mathcal{E}} \\
&= \lim_{n\to+\infty} \langle f_n, f_n \rangle + \mathcal{E}(f_n, f_n) \\
&= 0,
\end{aligned}$$

thus $f = 0$ and $\bar{\iota}$ is injective. Therefore, $\mathcal{H}_{\mathcal{E}}$ may be identified with a subspace of $\mathcal{H}$ and the quadratic form on $\mathcal{H}$ defined by

$$\bar{\mathcal{E}}(f) = \|f\|^2_{\mathcal{E}} - \|f\|^2, \quad f \in \mathcal{H}_{\mathcal{E}}$$

is closed because $(\mathcal{H}_{\mathcal{E}}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$ is a Hilbert space and obviously is an extension of $\mathcal{E}$. □

If a quadratic form $\mathcal{E}$ is closable, then its minimal closed extension is called the closure of $\mathcal{E}$. In that case, one can easily check that the closure of $\mathcal{E}$ is actually the quadratic form $\bar{\mathcal{E}}$ constructed in the previous proof.

Theorem 1.3.3. *Let $\mathcal{E}$ be a closed symmetric non negative bilinear form on $\mathcal{H}$. There exists a unique densely defined non positive self-adjoint operator A on $\mathcal{H}$ defined by*

$$\mathcal{D}(A) = \{ f \in \mathcal{F}, \exists g \in \mathcal{H}, \forall h \in \mathcal{F}, \mathcal{E}(f,h) = -\langle h, g \rangle \}$$

$$Af = g.$$

The operator A is called the generator of $\mathcal{E}$. Conversely, if A is a densely defined non positive self-adjoint operator on $\mathcal{H}$, one can define a closed symmetric non negative bilinear form $\mathcal{E}$ on $\mathcal{H}$ by

$$\mathcal{D}(\mathcal{E}) = \mathcal{D}((-A)^{1/2}), \quad \mathcal{E}(f,g) = \left\langle (-A)^{1/2} f, (-A)^{1/2} g \right\rangle .$$

Proof. Let $\mathcal{E}$ be a closed symmetric non negative bilinear form on $\mathcal{H}$. As usual, we denote by $\mathcal{F}$ the domain of $\mathcal{E}$. We note that for $\lambda > 0$, $\mathcal{F}$ equipped with the norm $(\|f\|^2 + \lambda \mathcal{E}(f))^{1/2}$ is a Hilbert space because $\mathcal{E}$ is closed. From the Riesz representation theorem, there exists then a linear operator $\mathbf{R}_\lambda : \mathcal{H} \to \mathcal{F}$ such that for every $f \in \mathcal{H}, g \in \mathcal{F}$

$$\langle f, g \rangle = \lambda \langle \mathbf{R}_\lambda f, g \rangle + \mathcal{E}(\mathbf{R}_\lambda f, g).$$

From the definition, the following properties are then easily checked:

1. $\|\mathbf{R}_\lambda f\| \le \frac{1}{\lambda} \|f\|$ (apply the definition of $\mathbf{R}_\lambda$ with $g = \mathbf{R}_\lambda f$ and then use the Cauchy-Schwarz inequality);
2. For every $f, g \in \mathcal{H}$, $\langle \mathbf{R}_\lambda f, g \rangle = \langle f, \mathbf{R}_\lambda g \rangle$;
3. $\mathbf{R}_{\lambda_1} - \mathbf{R}_{\lambda_2} + (\lambda_1 - \lambda_2) \mathbf{R}_{\lambda_1} \mathbf{R}_{\lambda_2} = 0$;
4. For every $f \in \mathcal{H}$, $\lim_{\lambda \to +\infty} \|\lambda \mathbf{R}_\lambda f - f\| = 0$.

We then claim that $\mathbf{R}_\lambda$ is invertible. Indeed, if $\mathbf{R}_\lambda f = 0$, then for $\alpha > \lambda$, one has from 3, $\mathbf{R}_\alpha f = 0$. Therefore $f = \lim_{\alpha \to +\infty} \mathbf{R}_\alpha f = 0$. Denote then

$$Af = \lambda f - \mathbf{R}_\lambda^{-1} f,$$

and $\mathcal{D}(A)$ is the range of $\mathbf{R}_\lambda$. It is straightforward to check that A does not depend on λ. The operator A is a densely defined self-adjoint operator that satisfies the properties stated in the theorem (Exercise !).
Conversely, if A is a densely defined non positive self-adjoint operator on $\mathcal{H}$, then $(-A)^{1/2}$ is a densely defined self-adjoint operator and the quadratic form

$$\mathcal{E}(f,g) := \left\langle (-A)^{1/2} f, (-A)^{1/2} g \right\rangle$$

is closed and densely defined on $\mathcal{D}((-A)^{1/2})$. □

Exercise 1.3.4. *Prove the properties 1,2,3,4 of the previous proof.*

In practice, the following lemma is often useful to construct closed quadratic forms.

Lemma 1.3.5. *Let A be a densely defined non positive symmetric operator $\mathcal{D}(A) \to \mathcal{H}$. The quadratic form*

$$\mathcal{E}(f,g) = -\langle f, Ag \rangle , \quad f, g \in \mathcal{D}(A)$$

is closable and the generator of its closure is a self-adjoint extension of A.

Proof. This follows from the lemma 1.3.2. □

1.4 Semigroups and quadratic forms

Theorem 1.4.1. *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $\mathcal{H}$. One can define a closed quadratic form on $\mathcal{H}$ by*

$$\mathcal{E}(f,f) := \lim_{t\to 0} \left\langle \frac{\mathrm{Id} - P_t}{t} f, f \right\rangle,$$

where the domain of this form is the set of f's for which the limit exists. The quadratic form $\mathcal{E}$ is called the quadratic form associated to the semigroup $(P_t)_{t\geq 0}$.

Proof. Let A be the generator of the semigroup $(P_t)_{t\geq 0}$. We use spectral theorem to represent A as

$$U^{-1}AUg(x) = -\lambda(x)g(x),$$

so that

$$U^{-1}P_tUg(x) = e^{-t\lambda(x)}g(x).$$

We then note that for every $g \in L^2(\Omega,\nu)$,

$$\left\langle \frac{\mathrm{Id} - P_t}{t} Ug, Ug \right\rangle = \int_\Omega \frac{1-e^{-t\lambda(x)}}{t} g(x)^2 d\nu(x).$$

This proves that for every $f \in \mathcal{H}$, the map $t \to \left\langle \frac{\mathrm{Id}-P_t}{t} f, f \right\rangle$ is non increasing. Therefore, the limit $\lim_{t\to 0} \left\langle \frac{\mathrm{Id}-P_t}{t} f, f \right\rangle$ exists if and only if $\int_\Omega (U^{-1}f)^2(x)\lambda(x)d\nu(x) < +\infty$, which is equivalent to the fact that $f \in \mathcal{D}((-A)^{1/2})$. In which case we have

$$\lim_{t\to 0} \left\langle \frac{\mathrm{Id} - P_t}{t} f, f \right\rangle = \|(-A)^{1/2}f\|^2.$$

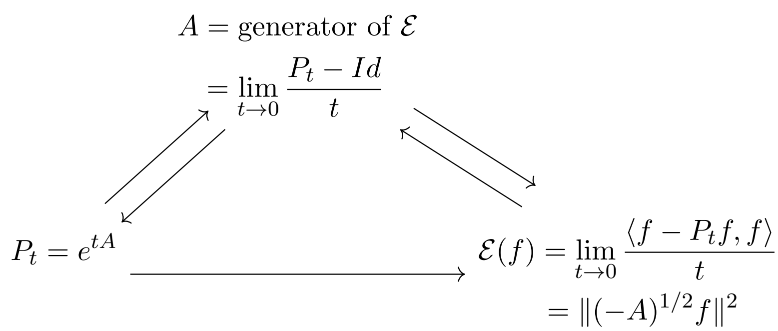
Since $(-A)^{1/2}$ is a densely defined self-adjoint operator, the quadratic form

$$\mathcal{E}(f) := \|(-A)^{1/2}f\|^2$$

is closed and densely defined on $\mathcal{F} := \mathcal{D}((-A)^{1/2})$. □

1.5 Summary: The golden triangle

As a conclusion, one has bijections between the set of non positive self-adjoint operators, the set of closed symmetric non negative bilinear form and the set of strongly continuous self-adjoint contraction semigroups. This is the golden triangle of the theory of heat semigroups on Hilbert spaces !



1.6 A first example: The Dirichlet energy on an open set $\Omega \subset \mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be an open connected set. We do not assume any regularity on the boundary of Ω. Classically, one can define the $(1,2)$ Sobolev space

$$W^{1,2}(\Omega) = \left\{ f \in L^2(\Omega) : \frac{\partial f}{\partial x_i} \in L^2(\Omega) \right\}$$

where the derivatives $\frac{\partial u}{\partial x_i}$ are understood in the weak sense. The quadratic form

$$\mathcal{E}(f,g) = \int_\Omega \langle \nabla f, \nabla g \rangle \, dx = \sum_{i=1}^n \int_\Omega \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} dx$$

with domain $W^{1,2}(\Omega)$ is then a closed densely defined quadratic form on $L^2(\Omega)$ since it is well-known that the Sobolev norm

$$\|f\|^2_{W^{1,2}(\Omega)} = \|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)}$$

is complete. The generator of the form $\mathcal{E}$ is called the Neumann Laplacian on Ω.
On the other hand, let

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

be the usual Laplacian on $\mathbb{R}^n$, the derivatives being understood in the ordinary sense, and $C_c^\infty(\Omega)$ be the set of smooth functions with a compact support included in Ω. Then, from the lemma 1.3.5, the quadratic form

$$\mathcal{E}_0(f,g) = -\int_\Omega f \Delta g dx$$

with domain $C_c^\infty(\Omega)$ is closable. The domain of the closure of $\mathcal{E}_0$ is the Sobolev space $W_0^{1,2}(\Omega)$ and the generator of the closure of $\mathcal{E}_0$ is called the Dirichlet Laplacian on Ω.

Notice that both the Neumann and the Dirichlet Laplacian are self-adjoint extensions of the Laplacian Δ with domain $C_c^\infty(\Omega)$. In general, the Neumann and Dirichlet Laplacian do not coincide. For instance if the boundary of Ω is smooth, then smooth functions in the domain of the Neumann Laplacian have vanishing normal derivatives while smooth functions in the domain of the Dirichlet Laplacian vanish on the boundary of u.

Chapter 2

Markovian semigroups and Dirichlet forms

Let $(X, \mathcal{B})$ be a measurable space. We say that $(X, \mathcal{B})$ is a *good* measurable space if there is a countable family generating $\mathcal{B}$ and if every finite measure γ on $(X \times X, \mathcal{B} \otimes \mathcal{B})$ can be decomposed as

$$\gamma(dxdy) = k(x, dy)\gamma_1(dx)$$

where γ_1 is the projection of γ on the first coordinate and k is a kernel, i.e $k(x, \cdot)$ is a finite measure on $(X, \mathcal{B})$ and $x \to k(x, A)$ is measurable for every $A \in \mathcal{B}$.
For instance, if X is a Polish space (or a Radon space) equipped with its Borel σ-field, then it is a good measurable space.
Throughout the chapter, we will consider $(X, \mathcal{B}, \mu)$ to be a good measurable space equipped with a σ-finite measure μ.

2.1 Markovian semigroups

Definition 2.1.1. *Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $L^2(X, \mu)$. The semigroup $(P_t)_{t \geq 0}$ is called Markovian if and only if for every $f \in L^2(X, \mu)$ and $t \geq 0$:*

1. $$f \geq 0, \ a.e \implies P_t f \geq 0, \ a.e \,.$$

2. $$f \leq 1, \ a.e \implies P_t f \leq 1, \ a.e \,.$$

We note that if $(P_t)_{t \geq 0}$ is Markovian, then for every $f \in L^2(X, \mu) \cap L^\infty(X, \mu)$,

$$\|P_t f\|_{L^\infty(X,\mu)} \leq \|f\|_{L^\infty(X,\mu)}.$$

As a consequence $(P_t)_{t \geq 0}$ can be extended to a contraction semigroup defined on all of $L^\infty(X, \mu)$.

Definition 2.1.2. *A transition function $\{p_t, t \geq 0\}$ on X is a family of kernels*

$$p_t : X \times \mathcal{B} \to [0,1]$$

such that:

1. *For $t \geq 0$ and $x \in X$, $p_t(x, \cdot)$ is a finite measure on X;*
2. *For $t \geq 0$ and $A \in \mathcal{B}$ the application $x \to p_t(x, A)$ is measurable;*
3. *For $s, t \geq 0$, a.e. $x \in X$ and $A \in \mathcal{B}$,*

$$p_{t+s}(x, A) = \int_X p_t(y, A) p_s(x, dy). \tag{2.1.1}$$

The relation (2.1.1) is often called the Chapman-Kolmogorov relation

Theorem 2.1.3 (Heat kernel measure)**.** *Let $(P_t)_{t\geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. There exists a transition function $\{p_t, t \geq 0\}$ on X such that for every $f \in L^\infty(X, \mu)$ and a.e. $x \in X$*

$$P_t f(x) = \int_X f(y) p_t(x, dy), \quad t > 0. \tag{2.1.2}$$

This transition function is called the heat kernel measure associated to $(P_t)_{t\geq 0}$.

Remark 2.1.4. *In general, the heat kernel measures are sub-probability measures i.e. satisfy $0 \leq p_t(x, X) \leq 1$. If they are probability measures, i.e. $P_t 1 = 1$ for every $t \geq 0$, the semigroup is said to be stochastically complete.*

The proof relies on the following lemma sometimes called the bi-measure theorem. A set function $\nu : \mathcal{B} \otimes \mathcal{B} \to [0, +\infty)$ is called a bi-measure, if for every $A \in \mathcal{B}$, $\nu(A, \cdot)$ and $\nu(\cdot, A)$ are measures.

Lemma 2.1.5. *If $\nu : \mathcal{B} \otimes \mathcal{B} \to [0, +\infty)$ is a bi-measure, then there exists a measure γ on $\mathcal{B} \otimes \mathcal{B}$ such that for every $A, B \in \mathcal{B}$,*

$$\gamma(A \times B) = \nu(A, B).$$

Proof of Theorem 2.1.3. We assume that μ is finite and let as an exercise the extension to σ-finite measures. For $t > 0$, we consider the set function

$$\nu_t(A, B) = \int_X 1_A P_t 1_B d\mu.$$

Since P_t is supposed to be Markovian, it is a bi-measure. From the bi-measure theorem, there exists a measure γ_t on $\mathcal{B} \otimes \mathcal{B}$ such that for every $A, B \in \mathcal{B}$,

$$\gamma_t(A \times B) = \nu_t(A, B) = \int_X 1_A P_t 1_B d\mu.$$

The projection of γ_t on the first coordinate is $(P_t 1)d\mu$, thus from the measure decomposition theorem, γ_t can be decomposed as

$$\gamma_t(dxdy) = p_t(x, dy)\mu(dx)$$

for some kernel p_t. One has then for every $A, B \in \mathcal{B}$

$$\int_X 1_A P_t 1_B d\mu = \int_A \int_B p_t(x, dy)\mu(dx),$$

from which it follows that for every $f \in L^\infty(X, \mu)$, and a.e. $x \in X$

$$P_t f(x) = \int_X f(y) p_t(x, dy).$$

The relation

$$p_{t+s}(x, A) = \int_X p_t(y, A) p_s(x, dy)$$

follows from the semigroup property. □

Exercise 2.1.6. *Prove Theorem 2.1.3 if μ is σ-finite.*

Exercise 2.1.7. *Show that for every non-negative measurable function $F : X \times X \to \mathbb{R}$,*

$$\int_X \int_X F(x, y) p_t(x, dy) d\mu(x) = \int_X \int_X F(x, y) p_t(y, dx) d\mu(y). \tag{2.1.3}$$

Definition 2.1.8. *Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. We say that the semigroup $\{P_t\}_{t \in [0,\infty)}$ admits a heat kernel if the heat kernel measures have a density with respect to μ, i.e. there exists a measurable function $p : \mathbb{R}_{>0} \times X \times X \to \mathbb{R}_{\geq 0}$, such that for every $t > 0$, a.e.$x, y \in X$, $f \in L^\infty(X, \mu)$,*

$$P_t f(x) = \int_X p_t(x, y) f(y) d\mu(y).$$

If the heat kernel exists, we will sometimes denote $p(t, x, y)$ as $p_t(x, y)$ for $t > 0$ and a.e. $x, y \in X$.

2.2 Dirichlet forms

Definition 2.2.1. *A function v on X is called a normal contraction of the function u if for almost every $x, y \in X$,*

$$|v(x) - v(y)| \leq |u(x) - u(y)| \text{ and } |v(x)| \leq |u(x)|.$$

Definition 2.2.2. *Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a densely defined closed quadratic form on $L^2(X, \mu)$. The form $\mathcal{E}$ is called a Dirichlet form if it is Markovian, that is, has the property that if $u \in \mathcal{F}$ and v is a normal contraction of u then $v \in \mathcal{F}$ and*

$$\mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

Exercise 2.2.3. *Show that a densely defined closed quadratic form on $L^2(X, \mu)$ is Markovian if and only if for every $u \in \mathcal{F}$, $(0 \vee u) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}((0 \vee u) \wedge 1, (0 \vee u) \wedge 1) \leq \mathcal{E}(u, u)$.*

Theorem 2.2.4. *Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction semigroup on $L^2(X, \mu)$. Then, $(P_t)_{t \geq 0}$ is a Markovian semigroup if and only if the associated closed symmetric form on $L^2(X, \mu)$ is a Dirichlet form.*

Proof. Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. There exists a transition function $\{p_t, t \geq 0\}$ on X such that for every $u \in L^\infty(X, \mu)$ and a.e. $x \in X$

$$P_t u(x) = \int_X u(y) p_t(x, dy), \quad t > 0.$$

Denote

$$k_t(x) = P_t 1(x) = \int_X p_t(x, dy).$$

We observe that from the Markovian property of P_t, we have $0 \leq k_t \leq 1$ a.e. We have then

$$\frac{1}{2} \int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x) = \int_X u(x)^2 k_t(x) d\mu(x) - \int_X u(x) P_t u(x) d\mu(x).$$

Therefore,

$$\langle u - P_t u, u \rangle = \frac{1}{2} \int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x) + \int_X u(x)^2 (1 - k_t(x)) d\mu(x).$$

Let us now assume that $u \in \mathcal{F}$ and that v is a normal contraction of u. One has

$$\int_X \int_X (v(x) - v(y))^2 p_t(x, dy) d\mu(x) \leq \int_X \int_X (u(x) - u(y))^2 p_t(x, dy) d\mu(x)$$

and

$$\int_X v(x)^2 (1 - k_t(x)) d\mu(x) \leq \int_X u(x)^2 (1 - k_t(x)) d\mu(x).$$

Therefore,

$$\langle v - P_t v, v \rangle \leq \langle u - P_t u, u \rangle$$

Since $u \in \mathcal{F}$, one knows that $\frac{1}{t} \langle u - P_t u, u \rangle$ converges to $\mathcal{E}(u)$ when $t \to 0$. Since $\frac{1}{t} \langle v - P_t v, v \rangle$ is non-increasing and bounded it does converge when $t \to 0$. Thus $v \in \mathcal{F}$ and

$$\mathcal{E}(v) \leq \mathcal{E}(u).$$

One concludes that $\mathcal{E}$ is Markovian.

Now, consider a Dirichlet form $\mathcal{E}$ and denote by P_t the associated semigroup in $L^2(X, \mu)$ and by A its generator. The main idea is to first prove that for $\lambda > 0$, the resolvent operator $(\lambda \mathrm{Id} - A)^{-1}$ preserves the positivity of function. Then, we may conclude by the fact that for $f \in L^2(X,\mu)$, in the $L^2(X,\mu)$ sense

$$P_t f = \lim_{n \to +\infty} \left(\mathrm{Id} - \frac{t}{n} L \right)^{-n} f.$$

Let $\lambda > 0$. We consider on $\mathcal{F}$ the norm

$$\|f\|_\lambda^2 = \|f\|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(f,f)$$

From the Markovian property of $\mathcal{E}$, if $u \in \mathcal{F}$, then $|u| \in \mathcal{F}$ and

$$\mathcal{E}(|u|,|u|) \le \mathcal{E}(u,u). \tag{2.2.1}$$

We consider the bounded operator

$$\mathbf{R}_\lambda = (\mathrm{Id} - \lambda A)^{-1}$$

that goes from $L^2(X,\mu)$ to $\mathcal{D}(A) \subset \mathcal{F}$. For $f \in \mathcal{F}$ and $g \in L^2(X,\mu)$ with $g \ge 0$, we have

$$\begin{aligned} \langle |f|, \mathbf{R}_\lambda g \rangle_\lambda &= \langle |f|, \mathbf{R}_\lambda g \rangle_{L^2(X,\mu)} - \lambda \langle |f|, A\mathbf{R}_\lambda g \rangle_{L^2(X,\mu)} \\ &= \langle |f|, (\mathrm{Id} - \lambda A)\mathbf{R}_\lambda g \rangle_{L^2(X,\mu)} \\ &= \langle |f|, g \rangle_{L^2(X,\mu)} \\ &\ge |\langle f, g \rangle_{L^2(X,\mu)}| \\ &\ge |\langle f, \mathbf{R}_\lambda g \rangle_\lambda|. \end{aligned}$$

Moreover, from inequality (2.2.1), for $f \in \mathcal{F}$,

$$\begin{aligned} \| \, |f| \, \|_\lambda^2 &= \| \, |f| \, \|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(|f|,|f|) \\ &\le \|f\|_{L^2(X,\mu)}^2 + \lambda \mathcal{E}(f,f) \\ &\le \|f\|_\lambda^2. \end{aligned}$$

By taking $f = \mathbf{R}_\lambda g$ in the two above sets of inequalities, we draw the conclusion

$$|\langle \mathbf{R}_\lambda g, \mathbf{R}_\lambda g \rangle_\lambda| \le \langle |\mathbf{R}_\lambda g|, \mathbf{R}_\lambda g \rangle_\lambda \le \| \, |\mathbf{R}_\lambda g| \, \|_\lambda \|\mathbf{R}_\lambda g\|_\lambda \le |\langle \mathbf{R}_\lambda g, \mathbf{R}_\lambda g \rangle_\lambda|.$$

The above inequalities are therefore equalities which implies

$$\mathbf{R}_\lambda g = |\mathbf{R}_\lambda g|.$$

As a conclusion if $g \in L^2(X,\mu)$ is a.e. ≥ 0, then for every $\lambda > 0$, $(\mathrm{Id} - \lambda A)^{-1} g \ge 0$ a.e.. Thanks to the spectral theorem, in $L^2(X,\mu)$,

$$P_t g = \lim_{n \to +\infty} \left(\mathrm{Id} - \frac{t}{n} A \right)^{-n} g.$$

By passing to a subsequence that converges pointwise almost surely, we deduce that $P_t g \ge 0$ almost surely. The proof of

$$f \le 1, \text{ a.e} \implies P_t f \le 1, \text{ a.e} .$$

follows the same lines:

- The first step is to observe that if $0 \leq f \in \mathcal{F}$, then $1 \wedge f \in \mathcal{F}$ and moreover

$$\mathcal{E}(1 \wedge f, 1 \wedge f) \leq \mathcal{E}(f, f).$$

- Let $f \in L^2(X, \mu)$ satisfy $0 \leq f \leq 1$ and set $g = \mathbf{R}_\lambda f = (\mathrm{Id} - \lambda A)^{-1} f \in \mathcal{F}$ and $h = 1 \wedge g$. According to the first step, $h \in \mathcal{F}$ and $\mathcal{E}(h, h) \leq \mathcal{E}(g, g)$. Now, we observe that:

$$\begin{aligned}
&\|g - h\|_\lambda^2 \\
=&\|g\|_\lambda^2 - 2\langle g, h\rangle_\lambda + \|h\|_\lambda^2 \\
=&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - 2\langle f, h\rangle_{L^2(X,\mu)} + \|h\|^2_{L^2(X,\mu)} + \lambda\mathcal{E}(h, h) \\
=&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - \|f\|^2_{L^2(X,\mu)} + \|f - h\|^2_{L^2(X,\mu)} + \lambda\mathcal{E}(h, h) \\
\leq&\langle \mathbf{R}_\lambda f, f\rangle_{L^2(X,\mu)} - \|f\|^2_{L^2(X,\mu)} + \|f - g\|^2_{L^2(X,\mu)} + \lambda\mathcal{E}(g, g) = 0.
\end{aligned}$$

  As a consequence $g = h$, that is $0 \leq g \leq 1$.

- The previous step shows that if $f \in L^2(X, \mu)$ satisfies $0 \leq f \leq 1$ then for every $\lambda > 0$, $0 \leq (\mathrm{Id} - \lambda L)^{-1} f \leq 1$. Thanks to spectral theorem, in $L^2(X, \mu)$,

$$\mathbf{P}_t f = \lim_{n \to +\infty} \left(\mathrm{Id} - \frac{t}{n} L\right)^{-n} f.$$

  By passing to a subsequence that converges pointwise almost surely, we deduce that $0 \leq \mathbf{P}_t f \leq 1$ almost surely.

□

2.3 The L^p theory of heat semigroups

Our goal, in this section, is to define, for $1 \leq p \leq +\infty$, P_t on $L^p := L^p(X, \mu)$. This may be done in a natural way by using the Riesz-Thorin interpolation theorem that we recall below.

Theorem 2.3.1 (Riesz-Thorin interpolation theorem)**.** *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and $\theta \in (0, 1)$. Define $1 \leq p, q \leq \infty$ by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T is a linear map such that

$$T : L^{p_0} \to L^{q_0}, \quad \|T\|_{L^{p_0} \to L^{q_0}} = M_0$$

$$T : L^{p_1} \to L^{q_1}, \quad \|T\|_{L^{p_1} \to L^{q_1}} = M_1,$$

then, for every $f \in L^{p_0} \cap L^{p_1}$,

$$\|Tf\|_q \leq M_0^{1-\theta} M_1^{\theta} \|f\|_p.$$

Hence T extends uniquely as a bounded map from L^p to L^q with

$$\|T\|_{L^p \to L^q} \leq M_0^{1-\theta} M_1^{\theta}.$$

Remark 2.3.2. *The statement that T is a linear map such that*

$$T : L^{p_0} \to L^{q_0}, \quad \|T\|_{L^{p_0} \to L^{q_0}} = M_0$$

$$T : L^{p_1} \to L^{q_1}, \quad \|T\|_{L^{p_1} \to L^{q_1}} = M_1,$$

means that there exists a map $T : L^{p_0} \cap L^{p_1} \to L^{q_0} \cap L^{q_1}$ with

$$\sup_{f \in L^{p_0} \cap L^{p_1}, \|f\|_{p_0} \leq 1} \|Tf\|_{q_0} = M_0$$

and

$$\sup_{f \in L^{p_0} \cap L^{p_1}, \|f\|_{p_1} \leq 1} \|Tf\|_{q_1} = M_1.$$

In such a case, T can be uniquely extended to bounded linear maps $T_0 : L^{p_0} \to L^{q_0}$, $T_1 : L^{p_1} \to L^{q_1}$. With a slight abuse of notation, these two maps are both denoted by T in the theorem.

Remark 2.3.3. *If $f \in L^{p_0} \cap L^{p_1}$ and p is defined by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then by Hölder's inequality, $f \in L^p$ and*

$$\|f\|_p \leq \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}.$$

We now are in position to state the following theorem:

Theorem 2.3.4. *Let $(P_t)_{t \geq 0}$ be a strongly continuous self-adjoint contraction Markovian semigroup on $L^2(X, \mu)$. The space $L^1 \cap L^\infty$ is invariant under P_t and P_t may be extended from $L^1 \cap L^\infty$ to a contraction semigroup $(P_t^{(p)})_{t \geq 0}$ on L^p for all $1 \leq p \leq \infty$: For $f \in L^p$,*

$$\|P_t f\|_{L^p} \leq \|f\|_{L^p}.$$

These semigroups are consistent in the sense that for $f \in L^p \cap L^q$,

$$P_t^{(p)} f = P_t^{(q)} f.$$

Proof. If $f, g \in L^1 \cap L^\infty$ which is a subset of $L^1 \cap L^\infty$, then,

$$\begin{aligned} \left| \int_X (P_t f) g d\mu \right| &= \left| \int_X f (P_t g) d\mu \right| \\ &\leq \|f\|_{L^1} \|P_t g\|_{L^\infty} \\ &\leq \|f\|_{L^1} \|g\|_{L^\infty}. \end{aligned}$$

This implies

$$\|P_t f\|_{L^1} \leq \|f\|_{L^1}.$$

The conclusion follows then from the Riesz-Thorin interpolation theorem. □

Exercise 2.3.5. *Show that if $f \in L^p$ and $g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ then,*

$$\int_{\mathbb{R}^n} f P_t^{(q)} g d\mu = \int_{\mathbb{R}^n} g P_t^{(p)} f d\mu.$$

Exercise 2.3.6.

1. *Show that for each $f \in L^1$, the L^1-valued map $t \to P_t^{(1)} f$ is continuous.*
2. *Show that for each $f \in L^p$, $1 < p < 2$, the L^p-valued map $t \to P_t^{(p)} f$ is continuous.*
3. *Finally, by using the reflexivity of L^p, show that for each $f \in L^p$ and every $p \geq 1$, the L^p-valued map $t \to P_t^{(p)} f$ is continuous.*

We mention, that in general, the L^∞ valued map $t \to P_t^{(\infty)} f$ is not continuous.

2.4 Diffusion operators as generators of Dirichlet forms

Consider a diffusion operator

$$L = \sum_{i,j=1}^{n} \sigma_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},$$

where b_i and σ_{ij} are continuous functions on $\mathbb{R}^n$ and for every $x \in \mathbb{R}^n$, the matrix $(\sigma_{ij}(x))_{1 \leq i,j \leq n}$ is a symmetric and non negative matrix.
Assume that there is a measure μ on $\mathbb{R}^n$ which is equivalent to the Lebesgue measure with continuous density and that symmetrizes L in the sense that for every smooth and compactly supported functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} g L f d\mu = \int_{\mathbb{R}^n} f L g d\mu.$$

For instance, if one can write

$$Lf = -\mathbf{div}(a \nabla f),$$

where a is a smooth field of positive and symmetric matrices, then the Lebesgue measure symetrizes L. From the lemma 1.3.5 the quadratic form

$$\mathcal{E}(f,g) = -\int_{\mathbb{R}^n} g L f d\mu, \quad f, g \in C_c^\infty(\mathbb{R}^n)$$

is closable. Let $\bar{\mathcal{E}}$ denotes its closure in $L^2(\mathbb{R}^n, \mu)$.

Proposition 2.4.1. *The quadratic form $\bar{\mathcal{E}}$ is a Dirichlet form.*

Proof. We need to prove that $\bar{\mathcal{E}}$ is Markovian. It is enough to prove that if $u \in \mathcal{F} = \mathcal{D}(\mathcal{E})$, then $|u| \in \mathcal{F}$ with $\bar{\mathcal{E}}(|u|,|u|) \le \bar{\mathcal{E}}(u,u)$ and that if $u \in \mathcal{F}$ with $u \ge 0$, then $u \wedge 1 \in \mathcal{F}$ with $\bar{\mathcal{E}}(|u|,|u|) \le \bar{\mathcal{E}}(u,u)$. We prove the first requirement, the second being established in a similar manner is let as an exercise to the reader.
Let $u \in C_c^\infty(\mathbb{R}^n)$ and consider

$$\phi_\varepsilon(x) = \sqrt{x^2+\varepsilon^2} - \varepsilon, \quad \varepsilon > 0.$$

One can check that $\phi_\varepsilon(u) \to |u|$ in $L^2(\mathbb{R}^n,\mu)$ and that $\phi_\varepsilon(u)$ is a Cauchy sequence for the norm

$$\|f\|^2_{\mathcal{F}} = \|f\|_{L^2(\mathbb{R}^n,\mu)} + \bar{\mathcal{E}}(f,f).$$

Since $\bar{\mathcal{E}}$ is closed this implies that $|u| \in \mathcal{F}$ and that $\phi_\varepsilon(u) \to |u|$ converges to u in the above norm. Now, using chain rule we see that for every smooth function $u \in C_c^\infty(\mathbb{R}^n)$,

$$L\phi_\varepsilon(u)(x) \ge \frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}} Lu(x).$$

Multiplying by $\phi_\varepsilon(u)$ and integrating we get

$$\mathcal{E}(\phi_\varepsilon(u), \phi_\varepsilon(u)) \le \mathcal{E}(u,u)$$

Taking the limit $\varepsilon \to 0$ yields

$$\bar{\mathcal{E}}(|u|,|u|) \le \bar{\mathcal{E}}(u,u)$$

The above inequality extends then to all $u \in \mathcal{F}$ by using the density of $C_c^\infty(\mathbb{R}^n)$ in the $\|\cdot\|_{\mathcal{F}}$ norm and the closedness of $\mathcal{E}$. □

Chapter 3

Some examples of Dirichlet spaces and heat kernel estimates

3.1 Riemannian manifolds

Let $(\mathbb{M}, g)$ be a n-dimensional Riemannian manifold with Riemannian volume measure μ and Riemannian distance d. We consider the quadratic form $\mathcal{E}$ on $\mathbb{M}$, which is obtained as the closure in $L^2(\mathbb{M}, \mu)$ of the quadratic form

$$\int_{\mathbb{M}} \langle \nabla f, \nabla g \rangle d\mu, \quad f, g \in C_c^\infty(\mathbb{M}). \tag{3.1.1}$$

The domain of the closure $\mathcal{E}$ is the Sobolev space $W_0^{1,2}(\mathbb{M})$ and its generator Δ is a self-adjoint extension of the Laplace-Beltrami operator. If the manifold is complete (which is equivalent to the metric space $(\mathbb{M}, d)$ being complete) then $\mathcal{E}$ is the unique closed extension of (3.1.1) and $W_0^{1,2}(\mathbb{M}) = W^{1,2}(\mathbb{M})$. If we assume further that the Ricci curvature of $\mathbb{M}$ is bounded from below then the domain of Δ is the Sobolev space $W^{2,2}(\mathbb{M})$. If we even assume further that the Ricci curvature of $\mathbb{M}$ is non negative, it is a well-known result by Li and Yau that the heat semigroup P_t admits a smooth heat kernel function $p_t(x, y)$ on $[0, \infty) \times \mathbb{M} \times \mathbb{M}$ for which there are constants $c_1, c_2, C > 0$ such that whenever $t > 0$ and $x, y \in X$,

$$\frac{1}{C} \frac{e^{-c_1 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \leq p_t(x, y) \leq C \frac{e^{-c_2 d(x,y)^2/t}}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}}.$$

3.2 Carnot groups

A Carnot group of step N is a simply connected Lie group $\mathbb{G}$ whose Lie algebra can be stratified as follows:

$$\mathfrak{g} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \text{ for } s > N.$$

From the above properties, Carnot groups are nilpotent. The number

$$Q = \sum_{i=1}^{N} i \dim \mathcal{V}_i$$

is called the homogeneous dimension of $\mathbb{G}$.
Let $V_1, ..., V_d$ be a basis of the vector space $\mathcal{V}_1$. The vectors V_i's can be seen as left invariant vector fields on $\mathbb{G}$. The left invariant sub-Laplacian on $\mathbb{G}$ is the operator:

$$L = \sum_{i=1}^{d} V_i^2.$$

It is hypoelliptic and essentially self-adjoint on the space of smooth and compactly supported function $f : \mathbb{G} \to \mathbb{R}$ with the respect to the Haar measure μ of $\mathbb{G}$. The heat semigroup $(P_t)_{t\geq 0}$ on $\mathbb{G}$, defined through the spectral theorem, is then seen to be a Markov semigroup. By hypoellipticity of L, this heat semigroup admits a heat kernel denoted by $p_t(g, g')$. It is then known that p_t satisfies the double-sided Gaussian bounds:

$$\frac{C^{-1}}{t^{Q/2}} \exp\left(-\frac{C_1 d(g, g')^2}{t}\right) \leq p_t(g, g') \leq \frac{C}{t^{Q/2}} \exp\left(-C_2 \frac{d(g, g')^2}{t}\right), \tag{3.2.1}$$

for some constants $C, C_1, C_2 > 0$. Here $d(g, g')$ denotes the Carnot-Carathéodory distance from g to g' on $\mathbb{G}$ which is defined by

$$d(g, g') = \sup\left\{ |f(g) - f(g')|, \quad \sum_{i=1}^{d} (V_i f)^2 \leq 1 \right\}.$$

3.3 Sierpiński gasket

A large class of examples for which Dirichlet form theory is useful is the class of p.c.f. fractals. For the sake of presentation we illustrate in detail the case of the Sierpiński gasket, which is one of the most popular examples of a p.c.f. fractal.
One of the classical ways to define the Sierpiński gasket is as follows: let $V_0 = \{p_1, p_2, p_3\}$ be a set of vertices of an equilateral triangle of side 1 in $\mathbb{C}$. Define

$$\mathfrak{f}_i(z) = \frac{z - p_i}{2} + p_i, \quad \text{for} \quad i = 1, 2, 3. \tag{3.3.1}$$

The Sierpiński gasket K (see Figure 3.1) is the unique non-empty compact subset in $\mathbb{C}$ such that

$$K = \bigcup_{i=1}^{3} \mathfrak{f}_i(K). \tag{3.3.2}$$

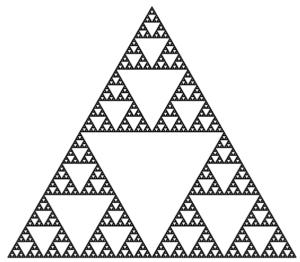


Figure 3.1: Sierpiński gasket.

The set V_0 is called the boundary of K, we will also denote it by ∂K. The Hausdorff dimension of K with respect to the Euclidean metric (denoted $d(x,y) = |x-y|$) is given by $d_h = \frac{\ln 3}{\ln 2}$. A (normalized) Hausdorff measure on K is given by the Borel measure μ on K such that for every $i_1, \ldots, i_n \in \{1,2,3\}$,

$$\mu\left(\mathfrak{f}_{i_1} \circ \cdots \circ \mathfrak{f}_{i_n}(K)\right) = 3^{-n}.$$

This measure μ is d_h-Ahlfors regular, i.e., there exist constants $c, C > 0$ such that for every $x \in K$ and $r \in [0, \operatorname{diam}(K)]$,

$$c r^{d_h} \le \mu(B(x,r)) \le C r^{d_h}. \tag{3.3.3}$$

It will be useful to approximate the gasket K by a sequence of discrete objects. Namely, starting from the set $V_0 = \{p_1, p_2, p_3\}$, we define a sequence of sets $\{V_m\}_{m\ge 0}$ inductively by

$$V_{m+1} = \bigcup_{i=1}^{3} \mathfrak{f}_i(V_m). \tag{3.3.4}$$

Then we have a natural sequence of Sierpiński gasket graphs (or pre-gaskets) $\{G_m\}_{m\ge 0}$ whose edges have length 2^{-m} and whose set of vertices is V_m, see Figure 3.2. Notice that $\#V_m = \frac{3(3^m+1)}{2}$. We will use the notations $V_* = \cup_{m\ge 0} V_m$ and $V_*^0 = \cup_{m\ge 0} V_m \setminus V_0$.
The Dirichlet form on the metric space K is defined by approximation. Let $m \ge 1$. For any $f \in \mathbb{R}^{V_m}$, we consider the quadratic form

$$\mathcal{E}_m(f,f) = \left(\frac{5}{3}\right)^m \sum_{p,q \in V_m, p \sim q} (f(p) - f(q))^2.$$

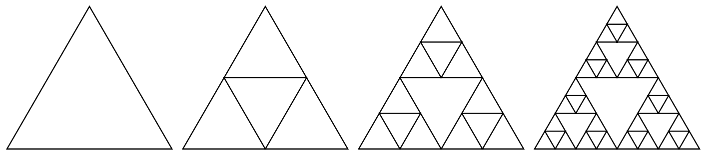


Figure 3.2: Sierpiński gasket graphs G_0, G_1, G_2 and G_3

We can then define a resistance form $(\mathcal{E}, \mathcal{F}_*)$ on V_* by setting

$$\mathcal{F}_* = \{f \in \mathbb{R}^{V_*}, \lim_{m\to\infty} \mathcal{E}_m(f,f) < \infty\}$$

and for $f \in \mathcal{F}_*$

$$\mathcal{E}(f,f) = \lim_{m\to\infty} \mathcal{E}_m(f,f).$$

Each function $f \in \mathcal{F}_*$ can be uniquely extended into a continuous function defined on K. We denote by $\mathcal{F}$ the set of functions with such extensions. $(\mathcal{E}, \mathcal{F})$ is a Dirichlet form on $L^2(K,\mu)$. The generator of the Dirichlet form $(\mathcal{E}, \mathcal{F})$, denoted by Δ, corresponds to the Laplacian with Neumann boundary condition.
In this example we have a continuous heat kernel $p_t(x,y)$ satisfying, for some $c_1, c_2, c_3, c_4 \in (0,\infty)$ and $d_H \geq 1, d_W \in [2,+\infty)$,

$$c_1 t^{-d_H/d_W} \exp\left(-c_2\left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right) \leq p_t(x,y) \leq c_3 t^{-d_H/d_W} \exp\left(-c_4\left(\frac{d(x,y)^{d_W}}{t}\right)^{\frac{1}{d_W-1}}\right)$$

for $\mu\times\mu$-a.e. $(x,y) \in X \times X$ and each $t \in \big(0,+\infty\big)$. Here, $d_W = \frac{\ln 5}{\ln 3}$ is the so-called walk dimension of the Sierpiński gasket.

3.4 Cheeger metric measure spaces

Consider a locally compact, complete, metric measure space (X,d,μ) where μ is a Radon measure. Any open metric ball centered at $x \in X$ with radius $r > 0$ will be denoted by

$$B(x,r) = \{y \in X, d(x,y) < r\}.$$

Definition 3.4.1. *The measure μ is said to be doubling if there exists a constant $C > 0$ such that for every $x \in X, r > 0$,*

$$0 < \mu(B(x,2r)) \leq C\mu(B(x,r)) < +\infty. \tag{VD}$$

The Lipschitz constant of a function $f \in \mathrm{Lip}(X)$ is defined as

$$(\mathrm{Lip} f)(y) := \limsup_{r\to 0^+} \sup_{x\in X, d(x,y)\leq r} \frac{|f(x)-f(y)|}{r}.$$

Definition 3.4.2. *The metric measure space* (X, d, μ) *is said to satisfy the 2-Poincaré inequality if for any* $f \in \mathrm{Lip}(X)$ *and any ball* $B(x, R)$ *of radius* $R > 0$*,*

$$\int_{B(x,R)} |f(y) - f_{B(x,R)}|^2 d\mu(y) \le CR^2 \int_{B(x,\lambda R)} (\mathrm{Lip}f)(y)^2 d\mu(y) \tag{P}$$

where

$$f_{B(x,R)} := \frac{1}{\mu(B(x,r))} \int_{B(x,R)} f(y) d\mu(y).$$

The constants $C > 0$ *and* $\lambda \ge 1$ *in* (P) *are independent from* x*,* R *and* f*.*

Definition 3.4.3. *A metric measure space satisfying* (VD) *and* (P) *is often called a Cheeger space (or PI space).*

One can construct a "nice" Dirichlet form and Laplacian on any Cheeger space by the using the technique of Γ-convergence.

Definition 3.4.4. *A sequence of forms* $\{\mathcal{E}_n\}_{n\ge1}$ *is said to Mosco-converge to* $\mathcal{E}$ *if*

1. *For any sequence* $\{f_n\}_{n\ge1} \subset L^2(X, \mu)$ *that converges* weakly *to* $f \in L^2(X, \mu)$ *in* $L^2(X, \mu)$*,*
$$\liminf_{n\to\infty} \mathcal{E}_n(f_n, f_n) \ge \mathcal{E}(f, f).$$

2. *For any* $f \in L^2(X, \mu)$ *there exists a sequence* $\{f_n\}_{n\ge1} \subset L^2(X, \mu)$ *that converges strongly to* f *in* $L^2(X, \mu)$ *and*
$$\limsup_{n\to\infty} \mathcal{E}(f_n, f_n) \le \mathcal{E}(f, f).$$

The idea is to consider Korevaar-Schoen type energy functionals defined for any $f \in L^2(X, \mu)$ as

$$E(f, r) := \int_X \frac{1}{\mu(B(x,r))} \int_{B(x,r)} \frac{|f(y) - f(x)|^2}{r^2} d\mu(y) d\mu(x), \tag{3.4.1}$$

and the associated Korevaar-Schoen space

$$KS^{1,2}(X) := \left\{ f \in L^2(X, \mu),\ \limsup_{r\to0^+} E(f, r) < +\infty \right\}.$$

One has then the following result:

Theorem 3.4.5. *There exists a Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(X, \mu)$ *such that:*

1. $\mathcal{E}$ *has domain* $\mathcal{F} = KS^{1,2}(X)$*;*

2. $\mathcal{E}$ *is a* Γ*-limit of* $E(f, r_n)$*, where* r_n *is a positive sequence such that* $r_n \to 0$*;*

3. $\mathcal{E}$ *has a continuous heat kernel* $p_t(x, y)$ *that satisfies for* $t > 0$ *and* $x, y \in X$*,*
$$\frac{1}{C} \frac{e^{-c_1 d(x,y)^2/t}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}} \le p_t(x, y) \le C \frac{e^{-c_2 d(x,y)^2/t}}{\sqrt{\mu(B(x,\sqrt{t}))\mu(B(y,\sqrt{t}))}}.$$

Chapter 4

Gagliardo-Nirenberg inequalities

Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a Dirichlet space and $\{P_t\}_{t\in[0,\infty)}$ denote the associated Markovian semigroup. Throughout the chapter, we shall assume that P_t admits a measurable heat kernel $p_t(x, y)$ satisfying, for some $C > 0$ and $\beta > 0$,

$$p_t(x, y) \leq Ct^{-\beta} \tag{4.0.1}$$

for $\mu \times \mu$-a.e. $(x, y) \in X \times X$, and for each $t \in \big(0, +\infty\big)$. We also assume stochastic completeness, which means that $P_t 1 = 1$ for every $t \geq 0$.
We use the techniques developed in [1] (see also the book [6]) to prove the Gagliardo-Nirenberg inequalities in that setting.

4.1 Preliminary lemmas

Lemma 4.1.1. *For every $f \in \mathcal{F}$, $t \geq 0$,*

$$\|P_t f - f\|_{L^2(X,\mu)} \leq C\sqrt{t}\mathcal{E}(f, f)^{1/2}.$$

Proof. Let Δ be the generator of the semigroup $(P_t)_{t\geq 0}$. We use spectral theorem to represent Δ as a multiplier in some $L^2(\Omega, \nu)$ space:

$$U^{-1}AUg(x) = -\lambda(x)g(x),$$

so that

$$\begin{aligned}
\|P_t f - f\|^2_{L^2(X,\mu)} &= \int_\Omega \Big(1 - e^{-t\lambda(x)}\Big)^2 (U^{-1}f)^2(x) d\nu(x) \\
&\leq Ct \int_\Omega \lambda(x)(U^{-1}f)^2(x) d\nu(x) \\
&= Ct\mathcal{E}(f, f).
\end{aligned}$$

□

Lemma 4.1.2. *For every $f \in \mathcal{F}$,*

$$\lim_{t \to 0} t^{-1} \int_X \int_X |f(x) - f(y)|^2 p_t(x,y) d\mu(x) d\mu(y) = 2\mathcal{E}(f,f).$$

Proof. Using $P_t 1 = 1$ we have

$$\begin{aligned}\int_X \int_X |f(x) - f(y)|^2 p_t(x,y) d\mu(x) d\mu(y) &= \int_X \int_X (f(x)^2 - 2f(x)f(y) + f(y)^2) p_t(x,y) d\mu(x) d\mu(y) \\ &= 2\int_X f(x)^2 d\mu(x) - 2\int_X f(x)(P_t f)(x) d\mu(x) \\ &= 2\int_X (f(x) - P_t f(x)) f(x) d\mu(x)\end{aligned}$$

and we conclude thanks to theorem 1.4.1. □

4.2 Sobolev inequality

Lemma 4.2.1. *Let $1 \le q < +\infty$. There exists a constant $C > 0$ such that for every $f \in \mathcal{F} \cap L^q(X,\mu)$ and $s \ge 0$,*

$$\sup_{s \ge 0} s^{1+\frac{q}{2\beta}} \mu\left(\{x \in X \,:\, |f(x)| > s\}\right)^{\frac{1}{2}} \le C\mathcal{E}(f,f)^{1/2} \|f\|_{L^q(X,\mu)}^{\frac{q}{2\beta}}.$$

Proof. Let $f \in \mathcal{F}$ and denote

$$F(s) = \mu\left(\{x \in X \,:\, |f(x)| > s\}\right).$$

We have then

$$F(s) \le \mu\left(\{x \in X \,:\, |f(x) - P_t f(x)| > s/2\}\right) + \mu\left(\{x \in X \,:\, |P_t f(x)| > s/2\}\right).$$

Now, from the heat kernel upper bound $p_t(x,y) \le Ct^{-\beta}$, $t > 0$, one deduces, for $g \in L^1(X,\mu)$, that $|P_t g(x)| \le Ct^{-\beta} \|g\|_{L^1(X,\mu)}$. Since P_t is a contraction in $L^\infty(X,\mu)$, by the Riesz-Thorin interpolation one obtains

$$|P_t f(x)| \le \frac{C^{1/q}}{t^{\beta/q}} \|f\|_{L^q(X,\mu)}.$$

Therefore, for $s = 2\frac{C^{1/q}}{t^{\frac{\beta}{q}}} \|f\|_{L^q(X,\mu)}$, one has $\mu\left(\{x \in X \,:\, |P_t f(x)| > s/2\}\right) = 0$. On the other hand, from lemma 4.1.1,

$$\mu\left(\{x \in X \,:\, |f(x) - P_t f(x)| > s/2\}\right) \le Cs^{-2} t\mathcal{E}(f,f).$$

We conclude that

$$F(s)^{1/2} \le Cs^{-1-\frac{q}{2\beta}} \mathcal{E}(f,f)^{1/2} \|f\|_{L^q(X,\mu)}^{\frac{q}{2\beta}}.$$

□

Lemma 4.2.2. *Assume $\beta > 1$. There exists a constant $C > 0$ such that for every $f \in \mathcal{F}$,*

$$\sup_{s \geq 0} s\, \mu\left(\{x \in X \,:\, |f(x)| \geq s\}\right)^{\frac{1}{q}} \leq C\mathcal{E}(f,f)^{1/2},$$

where $q = \frac{2\beta}{\beta-1}$.

Proof. Let $f \in \mathcal{F}$ be a non-negative function. For $k \in \mathbb{Z}$, we denote

$$f_k = (f - 2^k)_+ \wedge 2^k.$$

Observe that $f_k \in L^2(X,\mu)$ and $\|f_k\|_{L^2(X,\mu)} \leq \|f\|_{L^2(X,\mu)}$. Moreover, for every $x, y \in X$, $|f_k(x) - f_k(y)| \leq |f(x) - f(y)|$ and so $\mathcal{E}(f_k, f_k) \leq \mathcal{E}(f,f)$. We also note that $f_k \in L^1(X,\mu)$, with

$$\|f_k\|_{L^1(X,\mu)} = \int_X |f_k| d\mu \leq 2^k \mu(\{x \in X \,:\, f(x) \geq 2^k\}).$$

We now use lemma 4.2.1 to deduce:

$$\begin{aligned}\sup_{s \geq 0} s^{1+\frac{1}{2\beta}} \mu\left(\{x \in X \,:\, f_k(x) > s\}\right)^{\frac{1}{2}} &\leq C\mathcal{E}(f_k,f_k)^{1/2} \|f_k\|_{L^1(X,\mu)}^{\frac{1}{2\beta}} \\ &\leq C\mathcal{E}(f_k,f_k)^{1/2} \left(2^k \mu(\{x \in X \,:\, f(x) \geq 2^k\})\right)^{\frac{1}{2\beta}}.\end{aligned}$$

In particular, by choosing $s = 2^k$ we obtain

$$2^{k\left(1+\frac{1}{2\beta}\right)} \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right)^{\frac{1}{2}} \leq C\mathcal{E}(f_k,f_k)^{1/2} \left(2^k \mu(\{x \in X \,:\, f(x) \geq 2^k\})\right)^{\frac{1}{2\beta}}.$$

Let

$$M(f) = \sup_{k \in \mathbb{Z}} 2^k \mu(\{x \in X \,:\, f(x) \geq 2^k\})^{1/q},$$

where $q = \frac{2\beta}{\beta-1}$. Using the fact that $\frac{1}{q} = \frac{1}{2} - \frac{1}{2\beta}$ and the previous inequality we obtain:

$$2^k \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right)^{\frac{1}{2}} \leq C 2^{-\frac{kq}{2\beta}} \mathcal{E}(f,f)^{1/2} M(f)^{\frac{q}{2\beta}}.$$

and

$$2^k \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right)^{\frac{1}{q}} \leq C^{\frac{2}{q}} \mathcal{E}(f,f)^{1/q} M(f)^{\frac{1}{\beta}}.$$

Therefore

$$M(f)^{1-\frac{1}{\beta}} \leq 2C^{\frac{2}{q}} \mathcal{E}(f,f)^{1/q}$$

and one concludes

$$M(f) \leq 2^{q/2} C\mathcal{E}(f,f)^{1/2}.$$

This easily yields:

$$\sup_{s \geq 0} s\mu\left(\{x \in X \,:\, f(x) \geq s\}\right)^{\frac{1}{q}} \leq 2^{1+q/2} C\mathcal{E}(f,f)^{1/2}.$$

Let now $f \in \mathcal{F}$, which is not necessarily non-negative. From the previous inequality applied to $|f|$ we deduce

$$\sup_{s \geq 0} s\, \mu\left(\{x \in X \,:\, |f(x)| \geq s\}\right)^{\frac{1}{q}} \leq 2^{1+q/2} C \mathcal{E}(|f|, |f|)^{1/2} \leq 2^{1+q/2} C \mathcal{E}(f, f)^{1/2}. \qquad \square$$

Theorem 4.2.3. *Assume $\beta > 1$. There exists a constant $C > 0$ such that for every $f \in \mathcal{F}$,*

$$\|f\|_{L^q(X,\mu)} \leq C \mathcal{E}(f, f)^{1/2},$$

where $q = \frac{2\beta}{\beta - 1}$.

To show that the weak type inequality implies the desired Sobolev inequality, we will need another slicing argument and the following lemma is needed.

Lemma 4.2.4. *For $f \in \mathcal{F}$, $f \geq 0$, denote $f_k = (f - 2^k)_+ \wedge 2^k$, $k \in \mathbb{Z}$. There exists a constant $C > 0$ such that for every $f \in \mathcal{F}$,*

$$\sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k) \leq C \mathcal{E}(f, f).$$

Proof. Let $p_t(x, y)$ denote the heat kernel of the semigroup P_t. We first observe that, once we prove

$$\sum_{k \in \mathbb{Z}} \int_X \int_X |f_k(x) - f_k(y)|^2 p_t(x, y) d\mu(x) d\mu(y) \leq C \int_X \int_X |f(x) - f(y)|^2 p_t(x, y) d\mu(x) d\mu(y) \tag{4.2.1}$$

where $C > 0$ is independent from t, then

$$\begin{aligned} &\liminf_{t \to 0^+} \sum_{k \in \mathbb{Z}} t^{-1} \int_X \int_X |f_\rho(x) - f_\rho(y)|^2 p_t(x, y) d\mu(x) d\mu(y) \\ \leq &C \liminf_{t \to 0^+} t^{-1} \int_X \int_X |f(x) - f(y)|^p p_t(x, y) d\mu(x) d\mu(y), \end{aligned}$$

and, using the superadditivity of the lim inf, one concludes

$$\begin{aligned} &\sum_{k \in \mathbb{Z}} \liminf_{t \to 0^+} t^{-1} \int_X \int_X |f_k(x) - f_k(y)|^2 p_t(x, y) d\mu(x) d\mu(y) \\ \leq &C \liminf_{t \to 0^+} t^{-1} \int_X \int_X |f(x) - f(y)|^2 p_t(x, y) d\mu(x) d\mu(y) \end{aligned}$$

which from lemma 4.1.2 yields

$$\sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k) \leq C \mathcal{E}(f, f).$$

We therefore aim to prove the inequality (4.2.1). For each $k \in \mathbb{Z}$, set $B_k = \{x \in X : 2^k < f \leq 2^{k+1}\}$. In this way, the external integral on the left hand side of (4.2.1) is decomposed it into an integral over B_k and B_k^c. For the integrals over B_k, since the mapping $f \mapsto f_k$ is a contraction, it follows that

$$\sum_{k\in\mathbb{Z}} \int_{B_k} \int_X |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \le \int_X \int_X |f(x) - f(y)|^2 p_t(x,y) d\mu(x) d\mu(y). \tag{4.2.2}$$

To perform the integrals over B_k^c, we decompose them as

$$\sum_{k\in\mathbb{Z}} \int_{B_k^c} \int_{B_k} |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y) + \sum_{k\in\mathbb{Z}} \int_{B_k^c} \int_{B_k^c} |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y)$$
$$=: \sum_{k\in\mathbb{Z}} J_1(k) + \sum_{k\in\mathbb{Z}} J_2(k).$$

Again, the contraction property of $f \mapsto f_k$ yields

$$\sum_{k\in\mathbb{Z}} J_1(k) \le \sum_{k\in\mathbb{Z}} \int_X \int_{B_k} |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y)$$
$$\le \int_X \sum_{k\in\mathbb{Z}} \int_{B_k} |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \le \int_X \int_X |f(x) - f(y)|^2 p_t(x,y) d\mu(x) d\mu(y).$$

On the other hand, notice that for any $(x,y) \in B_k^c \times B_k^c$ we have $|f_k(x) - f_k(y)| \neq 0$ only if

$$(x,y) \in \{f(x) \le 2^k < f(y)/2\} \cup \{f(y) \le 2^k < f(x)/2\} =: Z_k \cup Z_k^*.$$

Also, $|f_k(x) - f_k(y)| = 2^k$ for $(x,y) \in Z_k \cup Z_k^*$. Thus,

$$\begin{aligned}\sum_{k\in\mathbb{Z}} J_2(k) &\le \sum_{k\in\mathbb{Z}} \int_X \int_X \big(\mathbf{1}_{Z_k}(x,y) + \mathbf{1}_{Z_k^*}(x,y)\big) |f_k(x) - f_k(y)|^2 p_t(x,y) d\mu(x) d\mu(y) \\ &= \int_X \int_X \sum_{k\in\mathbb{Z}} \big(\mathbf{1}_{Z_k}(x,y) + \mathbf{1}_{Z_k^*}(x,y)\big) 2^{2k} p_t(x,y) d\mu(x) d\mu(y).\end{aligned}$$

One can see that

$$\sum_{k\in\mathbb{Z}} \mathbf{1}_{Z_k}(x,y) 2^{2k} \le 2|f(x) - f(y)|^2$$

and the same holds for Z_k^*, hence

$$\sum_{k\in\mathbb{Z}} J_1(k) + \sum_{k\in\mathbb{Z}} J_2(k) \le 5 \int_X \int_X |f(x) - f(y)|^2 p_t(x,y) d\mu(x) d\mu(y).$$

Adding to these the term from (4.2.2) finally yields (4.2.1). □

We can now conclude the proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. Let $f \in \mathcal{F}$. We can assume $f \geq 0$. As before, denote $f_k = (f - 2^k)_+ \wedge 2^k$, $k \in \mathbb{Z}$. From Lemma 4.2.2 applied to f_k, we see that

$$\sup_{s \geq 0} s\mu\left(\{x \in X \,:\, |f_k(x)| \geq s\}\right)^{\frac{1}{q}} \leq C\mathcal{E}(f_k, f_k)^{1/2}.$$

In particular for $s = 2^k$, we get

$$2^k \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right)^{\frac{1}{q}} \leq C\mathcal{E}(f_k, f_k)^{1/2}.$$

Therefore,

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right) \leq C^q \sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k)^{q/2}.$$

Since $q \geq 2$, one has $\sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k)^{q/2} \leq \left(\sum_{k \in \mathbb{Z}} \mathcal{E}(f_k, f_k)\right)^{q/2}$. Thus, from the previous lemma

$$\sum_{k \in \mathbb{Z}} 2^{kq} \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right) \leq C\mathcal{E}(f, f)^{q/2}.$$

Finally, we observe that

$$\begin{aligned}\sum_{k \in \mathbb{Z}} 2^{kq} \mu\left(\{x \in X \,:\, f(x) \geq 2^{k+1}\}\right) &\geq \frac{q}{2^{q+1} - 2^q} \sum_{k \in \mathbb{Z}} \int_{2^{k+1}}^{2^{k+2}} s^{q-1} \mu\left(\{x \in X \,:\, f(x) \geq s\}\right) ds \\ &\geq \frac{1}{2^{q+1} - 2^q} \|f\|^q_{L^q(X,\mu)}.\end{aligned}$$

The proof is thus complete. □

4.3 Gagliardo-Nirenberg inequalities

In the general case $\beta > 0$ one can get from the lemma 4.2.1 the family of Gagliardo-Nirenberg inequalities.

Theorem 4.3.1. *Let $q = \frac{2\beta}{\beta - 1}$ with the convention that $q = \infty$ if $\beta = 1$. Let $r, s \in (0, +\infty]$ and $\theta \in (0, 1]$ satisfying*

$$\frac{1}{r} = \frac{\theta}{q} + \frac{1 - \theta}{s}.$$

If $\beta = 1$, we assume $r < +\infty$. Then, there exists a constant $C > 0$ such that for every $f \in \mathcal{F}$,

$$\|f\|_{L^r(X,\mu)} \leq C\mathcal{E}(f, f)^{\theta/2} \|f\|^{1-\theta}_{L^s(X,\mu)}. \tag{4.3.1}$$

We explicitly point out some particular cases of interest.

1. Assume that $\beta > 1$. If $r = s$, then $r = \frac{2\beta}{\beta - 1}$ and (4.3.1) recovers the Sobolev inequality

$$\|f\|_{L^r(X,\mu)} \leq C\mathcal{E}(f, f)^{1/2}.$$

2. Assume that $\beta > 1$. If $s = +\infty$ and $r \geq \frac{2\beta}{\beta-1}$, then (4.3.1) yields

$$\|f\|_{L^r(X,\mu)} \leq C\mathcal{E}(f,f)^{\theta/2}\|f\|_{L^\infty(X,\mu)}^{1-\theta}$$

   with $\theta = \frac{2\beta}{r(\beta-1)}$.

3. If $r = 2$ and $s = 1$, then (4.3.1) yields the Nash inequality

$$\|f\|_{L^2(X,\mu)} \leq C\mathcal{E}(f,f)^{\theta/2}\|f\|_{L^1(X,\mu)}^{1-\theta}$$

   with $\theta = \frac{\beta}{1+\beta}$.

In the case $\beta = 1$ one obtains the Trudinger-Moser inequalities.

Corollary 4.3.2. *Assume that $\beta = 1$. Then, there exist constants $c, C > 0$ such that for every $f \in \mathcal{F}$ with $\mathcal{E}(f,f) = 1$,*

$$\int_X \left(\exp\left(c|f|^2\right) - 1\right) d\mu \leq C\|f\|_{L^2(X,\mu)}^2.$$

Chapter 5

Further topics

In this chapter, let X be a locally compact and complete metric space equipped with a Radon measure μ supported on X. Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a Dirichlet form on X. We assume throughout that the heat semigroup P_t is stochastically complete, i.e. $P_t 1 = 1$ for every $t \geq 0$.

5.1 Regular Dirichlet forms, Energy measures

We denote by $C_c(X)$ the vector space of all continuous functions with compact support in X and $C_0(X)$ its closure with respect to the supremum norm.
A core for $(X, \mu, \mathcal{E}, \mathcal{F})$ is a subset $\mathcal{C}$ of $C_c(X) \cap \mathcal{F}$ which is dense in $C_c(X)$ in the supremum norm and dense in $\mathcal{F}$ in the norm

$$\left(\|f\|^2_{L^2(X,\mu)} + \mathcal{E}(f,f) \right)^{1/2}.$$

Definition 5.1.1. *The Dirichlet form $\mathcal{E}$ is called regular if it admits a core.*

Recall that for any $f, g \in \mathcal{F}$, we have

$$\mathcal{E}(f,g) = \lim_{t \to 0} \frac{1}{t} \langle (I - P_t) f, g \rangle = \lim_{t \to 0} \frac{1}{t} \int_X \int_X (f(x) - f(y)) g(x) p_t(x, dy) d\mu(x),$$

where $p_t(x, \cdot)$ are the heat kernel measures associated to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. From the symmetry property (2.1.3) of the heat kernel measure one also has

$$\mathcal{E}(f,g) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X (f(x) - f(y))(g(x) - g(y)) p_t(x, dy) d\mu(x).$$

Lemma 5.1.2. *For $f, g \in \mathcal{F} \cap L^\infty(X, \mu)$, $fg \in \mathcal{F}$ and*

$$\mathcal{E}(fg)^{1/2} \leq \|f\|_\infty \mathcal{E}(g)^{1/2} + \|g\|_\infty \mathcal{E}(f)^{1/2}$$

Proof. For $f, g \in \mathcal{F} \cap L^\infty(X, \mu)$ such that $fg \in \mathcal{F}$,

$$\mathcal{E}(fg) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X (f(x)g(x) - f(y)g(y))^2 p_t(x, dy) d\mu(x).$$

Write $f(x)g(x) - f(y)g(y) = f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))$, then by Minkowski's inequality

$$\begin{aligned}
&\left(\int_X \int_X (f(x)g(x) - f(y)g(y))^2 p_t(x, dy) d\mu(x) \right)^{1/2} \\
&\le \|f\|_\infty \left(\int_X \int_X (g(x) - g(y))^2 p_t(x, dy) d\mu(x) \right)^{1/2} + \|g\|_\infty \left(\int_X \int_X (f(x) - f(y))^2 p_t(x, dy) d\mu(x) \right)^{1/2}.
\end{aligned}$$

We conclude the expected inequality by multiplying $\frac{1}{\sqrt{2t}}$ and taking the limit $t \to 0$ for both sides above. □

Theorem 5.1.3 (Energy measures)**.** *Assume that $\mathcal{E}$ is regular. For $f \in \mathcal{F} \cap L^\infty(X, \mu)$, there exists a unique Radon measure on X, denoted by $d\Gamma(f)$, so that for every $\phi \in \mathcal{F} \cap C_c(X)$,*

$$\begin{aligned}
\int_X \phi \, d\Gamma(f) &= \frac{1}{2}[2\mathcal{E}(\phi f, f) - \mathcal{E}(\phi, f^2)] \\
&= \lim_{t \to 0} \frac{1}{2t} \int_X \int_X \phi(x)(f(x) - f(y))^2 p_t(x, dy) d\mu(x).
\end{aligned}$$

The Radon measure $d\Gamma(f)$ is called the energy measure of f (and is therefore the weak * limit of $\frac{1}{2t} \int_X (f(x) - f(y))^2 p_t(x, dy)$.)

Proof. Let $f \in \mathcal{F} \cap L^\infty(X, \mu)$. For any $\phi \in \mathcal{F} \cap C_c(X)$,

$$\frac{1}{2t} \int_X \int_X \phi(x)(f(x) - f(y))^2 p_t(x, dy) d\mu(x) = -\frac{1}{2t} \langle (I - P_t) f^2, \phi \rangle + \frac{1}{t} \langle (I - P_t) f, f\phi \rangle.$$

Letting $t \to 0$, the right hand side converges to $\frac{1}{2}[2\mathcal{E}(\phi f, f) - \mathcal{E}(\phi, f^2)]$.
On the other hand, observing that

$$\frac{1}{2t} \int_X \int_X |\phi(x)|(f(x) - f(y))^2 p_t(x, dy) d\mu(x) \le \|\phi\|_\infty \frac{1}{2t} \int_X \int_X (f(x) - f(y))^2 p_t(x, dy) d\mu(x),$$

we deduce

$$\left| \frac{1}{2}[2\mathcal{E}(\phi f, f) - \mathcal{E}(\phi, f^2)] \right| \le \|\phi\|_\infty \mathcal{E}(f).$$

Therefore we conclude the proof by applying the Riesz-Markov representation theorem. □

One can actually define $d\Gamma(f, f)$ for every $f \in \mathcal{F}$ using the following lemmas.

Lemma 5.1.4. *Let $f \in \mathcal{F}$. Then $f_n = \min(n, \max(-n, f)) \in \mathcal{F}$ and $\mathcal{E}(f - f_n) \to 0$.*

Proof. Let $f \in \mathcal{F}$. For every $x, y \in X$, we have

$$|f_n(x) - f_n(y)| \leq |f(x) - f(y)|$$

and $|f_n(x)| \leq |f(x)|$. So f_n is a normal contraction of f and $\mathcal{E}(f_n) \leq \mathcal{E}(f)$.
Recall that

$$\mathcal{E}(f - f_n) = \lim_{t \to 0} \frac{1}{2t} \int_X \int_X (f(x) - f_n(x) - f(y) + f_n(y))^2 p_t(x, dy) d\mu(x).$$

Expanding the square inside the integral gives that

$$\mathcal{E}(f - f_n) = \mathcal{E}(f) + \mathcal{E}(f_n) - 2\mathcal{E}(f_n, f) \leq 2\mathcal{E}(f) - 2\mathcal{E}(f_n, f)$$

Letting $n \to \infty$, we have $\mathcal{E}(f_n, f) \to \mathcal{E}(f)$. Therefore $\mathcal{E}(f - f_n)$ converges to 0 as $n \to \infty$. □

Lemma 5.1.5. *For $f, g \in \mathcal{F} \cap L^\infty(X, \mu)$ and nonnegative $\phi \in \mathcal{F} \cap C_c(X)$,*

$$\left| \sqrt{\int_X \phi \, d\Gamma(f)} - \sqrt{\int_X \phi \, d\Gamma(g)} \right|^2 \leq \int_X \phi \, d\Gamma(f - g) \leq \|\phi\|_{L^\infty(X,\mu)} \mathcal{E}(f - g)$$

Proof. The second inequality follows from the proof in Theorem 5.1.3. For the first inequality, it suffice to show that for any $f, g \in \mathcal{F} \cap L^\infty(X, \mu)$ and any nonnegative $\phi \in \mathcal{F} \cap C_c(X)$,

$$\sqrt{\int_X \phi \, d\Gamma(f)} \leq \sqrt{\int_X \phi \, d\Gamma(g)} + \sqrt{\int_X \phi \, d\Gamma(f - g)}.$$

Indeed, this inequality follow from similar proof as in Lemma 5.1.2 by noting that $f = f - g + g$ and using Minkowski's inequality. □

Thanks to the previous lemmas, by approximation, one can define $d\Gamma(f)$ for every $f \in \mathcal{F}$ by

$$d\Gamma(f) = \sup\{d\Gamma(f_n) : f_n = \min(n, \max(-n, f)), n = 1, 2, \cdots\}.$$

For $f, g \in \mathcal{F}$, one can define $d\Gamma(f, g)$ by polarization

$$d\Gamma(f, g) = \frac{1}{4} \left(d\Gamma(f + g) - d\Gamma(f - g) \right).$$

Theorem 5.1.6 (Beurling-Deny)**.** *Assume that $\mathcal{E}$ is regular. For $u, v \in \mathcal{F}$*

$$\mathcal{E}(u, v) = \int_X d\Gamma(u, v).$$

5.2 Hunt process associated with a regular Dirichlet form

Definition 5.2.1. *A Hunt process with state space X is a family of stochastic process $(X_t)_{t\geq 0}$ and probability measures $(\mathbb{P}_x)_{x\in X})$ defined on a measure space $(\Omega, \mathbf{F})$, such that $(X_t)_{t\geq 0}$ is adapted w.r.t. the right-continuous minimal completed admissible filtration $(\mathbf{F}_t)_{t\geq 0}$, $X_0 = x$, $\mathbb{P}_x - a.s.$ and the following hold:*

(i) *$x \to \mathbb{P}_x(X_t \in B)$ is measurable for all $t > 0$ and $B \in \mathcal{B}(X)$,*

(ii) *X is a strong Markov process, i.e. for every stopping time T, X_T is $\mathbf{F}_T$-measurable and for every $B \in \mathcal{B}(X)$*

$$\mathbb{P}_x(X_{T+t} \in B|\mathcal{F}_T) = \mathbb{P}_{X_T}(X_t \in B) \quad \mathbb{P}_x\text{-a.s. on } \{T < \infty\},$$

(iii) *X is right-continuous, i.e.*

$$\lim_{s\downarrow t} X_s = X_t, \quad \forall t \quad \mathbb{P}_x\text{-a.s.}$$

(iv) *X is quasi left-continuous, i.e. for all stopping times T and $(T_n)_n$ such that $T_n \uparrow T$ a.s.*

$$\lim_{n\to\infty} X_{T_n} = X_T, \quad \mathbb{P}_x\text{-a.s. on } \{T < \infty\}.$$

Remark 5.2.2.

(a) *Note that quasi left-continuity does not necessarily imply left-continuity, because the set*

$$A = \left\{\lim_{n\to\infty} X_{s_n} = X_t\right\}$$

might depend on the choice of sequence $(s_n)_n, s_n \uparrow t$.

(b) *In more generality one might consider situations where $(\mathbb{P}_x(X_t \in \cdot))_{x,t}$ are sub-probability measures (i.e. all the axioms of probability measures are satisfied but $\mathbb{P}_x(X_t \in \Omega) \leq 1$). In that case we can perform a one-point compactification of X by introducing a cemetery state $\partial \notin X$ and redefine $\mathbb{P}_x$ to be a probability measure on $X \cup \{\partial\}$.*

Theorem 5.2.3 (Fukushima)**.** *Assume that $\mathcal{E}$ is regular, then there exists a Hunt process $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in X})$ such that for μ-a.e. $x \in X$, $A \in \mathcal{B}(X)$ and $t \geq 0$,*

$$\mathbb{P}_x(X_t \in A) = p_t(x, A)$$

where $p_t(x, \cdot)$ are the heat kernel measures associated to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Proof. See Theorem 4.2.8 in [3]. □

5.3 Intrinsic metric

Definition 5.3.1. *The Dirichlet form is called* strongly local *if for any two functions $f, g \in \mathcal{F}$ with compact supports such that f is constant in a neighborhood of the support of g, we have $\mathcal{E}(f,g) = 0$.*

With respect to $\mathcal{E}$ we can define the following *intrinsic metric* $d_{\mathcal{E}}$ on X by

$$d_{\mathcal{E}}(x,y) = \sup\{u(x) - u(y) : u \in \mathcal{F} \cap C_0(X) \text{ and } d\Gamma(u,u) \leq d\mu\}. \tag{5.3.1}$$

Here the condition $d\Gamma(u,u) \leq d\mu$ means that $\Gamma(u,u)$ is absolutely continuous with respect to μ with Radon-Nikodym derivative bounded by 1.
The term "intrinsic metric" is potentially misleading because in general there is no reason why $d_{\mathcal{E}}$ is a metric on X (it could be infinite for a given pair of points x, y or zero for some distinct pair of points).

Definition 5.3.2. *A strongly local regular Dirichlet space is called strictly local if $d_{\mathcal{E}}$ is a metric on X and the topology induced by $d_{\mathcal{E}}$ coincides with the topology on X.*

Example 5.3.3 (Uniformly elliptic divergence form diffusion operators)**.** *On $\mathbb{R}^n$, we consider the divergence form operator*

$$Lf = -\mathbf{div}(\sigma \nabla f),$$

where σ is a smooth field of positive and symmetric matrices that satisfies

$$a\|x\|^2 \leq \langle x, \sigma(y)x\rangle \leq b\|x\|^2, \quad x, y \in \mathbb{R}^n,$$

for some constant $0 < a \leq b$. Consider the Dirichlet form

$$\mathcal{E}(f) = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \sigma_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx, \quad f \in W^{1,2}(\mathbb{R}^n).$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x,y) \simeq \|x - y\|.$$

Example 5.3.4 (Riemannian manifolds)**.** *Let $(\mathbb{M}, g)$ be a complete n-dimensional Riemannian manifold with Riemannian volume measure μ. We consider the standard Dirichlet form $\mathcal{E}$ on $\mathbb{M}$, which is obtained by closing the bilinear form*

$$\mathcal{E}(f,g) = \int_{\mathbb{M}} \langle \nabla f, \nabla g\rangle d\mu, \quad f, g \in C_0^\infty(\mathbb{M}).$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x,y) = d_g(x,y).$$

Example 5.3.5 (Carnot groups)**.** *Let $\mathbb{G}$ be a Carnot group with sub-Laplacian*

$$L = \sum_{i=1}^{d} V_i^2$$

and Dirichlet form

$$\mathcal{E}(f) = \int_{\mathbb{G}} \sum_{i=1} (V_i f)^2 d\mu.$$

Then $\mathcal{E}$ is a strictly local Dirichlet form such that

$$d_{\mathcal{E}}(x, y) = d_{CC}(x, y)$$

where d_{CC} is the so-called Carnot-Carathéodory distance which is defined as follows. An absolutely continuous curve $\gamma : [0, T] \to \mathbb{G}$ is said to be subunit for the operator L if for every smooth function $f : \mathbb{G} \to \mathbb{R}$ we have $\left|\frac{d}{dt} f(\gamma(t))\right| \le \sqrt{(\Gamma f)(\gamma(t))}$. We then define the subunit length of γ as $\ell_s(\gamma) = T$.
Given $x, y \in \mathbb{G}$, we indicate with

$$S(x, y) = \{\gamma : [0, T] \to \mathbb{G} \mid \gamma \textit{ is subunit for } \Gamma, \gamma(0) = x,\ \gamma(T) = y\}.$$

It is a consequence of the Chow-Rashevskii theorem that

$$S(x, y) \neq \emptyset, \qquad \textit{for every } x, y \in \mathbb{G}.$$

One defines then

$$d_{CC}(x, y) = \inf\{\ell_s(\gamma) \mid \gamma \in S(x, y)\}, \tag{5.3.2}$$

Example 5.3.6. *Consider on the Sierpinski gasket the standard Dirichlet form $\mathcal{E}$. Then $\mathcal{E}$ is regular, but unless f is constant, for $f \in \mathcal{F}$, $d\Gamma(f)$ is singular with respect to the Hausdorff measure μ, see [5]. As a consequence $\mathcal{E}$ is not strictly local.*

5.4 Further reading

Far from being exhaustive we mention the folllowing references for further reading:

(a) The book [3] is a standard comprehensive reference in the theory of Dirichlet forms and associated Hunt processes, see also [2].

(b) The book [4] shows how one can restrict Dirichlet forms to domains (abstract Dirichlet and Neumann boundary conditions).

(c) Parabolic regularity theory for the heat equation can be developed in the setting of abstract strictly local Dirichlet spaces, see [8, 7].

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